LOWER BOUNDS ON CIRCUITS

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Motivation

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Compact Circuits for Efficient Möbius Transform

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Abstract. The Möbius transform is a linear circuit used to compute the evaluations of a Boolean function over all points on its input domain. The operation is very useful in finding the solution of a system of polynomial equations over $GF(2)$ for obvious reasons. However the operation, although linear, needs exponential number of logic operations (around $n \cdot 2^{n-1}$ bit xors) for an n-variable Boolean function. As such, the

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dimensions then the Area(A) and Time(T) product of the FFT is subject to a lower bound $AT^2 \geq N^2$ (where $N = 2^n$) [Tho79]. Because the circuit has to hold the N input data, this implies that $AT \geq N^{3/2}$. This lower-bound is based on communication complexity, i.e. the fact that "wires take space". In this paper we begin with two circuits for Möbius Transform i.e. **Expmob1**/**Expmob2** that take $AT^2 = N \cdot \log(N)$ and $N \cdot (\log(N))^2$ respectively.

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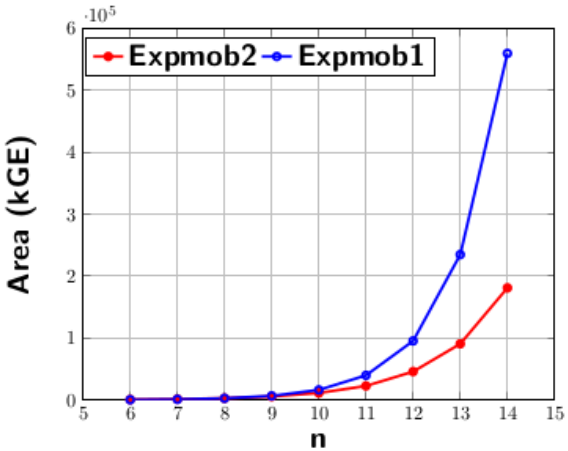
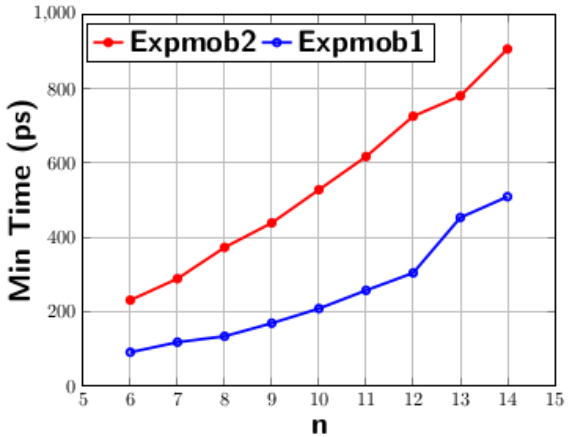
Expmob1 This architecture implements the circuit in Figure 2 as a single unrolled circuit, i.e. it implements all the n butterfly stages as dedicated circuits sequentially. Consider $\textbf{one}_i(x) : \{0,1\}^{n-1} \to \{0,1\}^n$ to be the function that inserts a 1 in the i-th MSB position of x, and $\textbf{zero}_i(x)$ to be a function that inserts a 0 in the same position, i.e $\textbf{one}_0(1001) = 1\ 1101$ and $\textbf{zero}_0(1001) = 0\ 1001$ etc. Note that there are a total of 2^{n-1} butterfly operations in each of the n stages. In the i-th stage (for $0 \leq i \leq n-1$), the j-th butterfly takes as input the bits in the position $\textbf{zero}_i(j)$ and $\textbf{one}_i(j)$ for all $0 \leq j \leq 2^{n-1} - 1$. This requires a total of $n \cdot 2^{n-1}$ number of 2-input xor gates in total. However such a circuit is able to compute the transformation in a single cycle.

Motivation

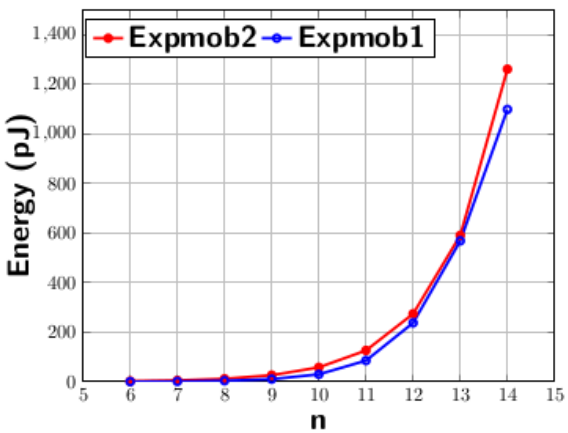
Compact Circuits for Efficient Möbius Transform

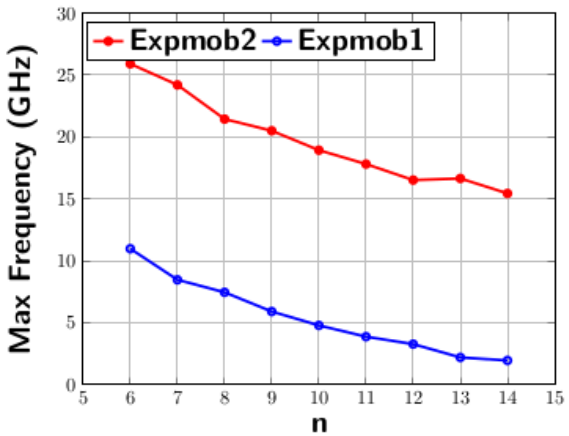


(a) Area



(b) Time





Bound from Comunication Complexity

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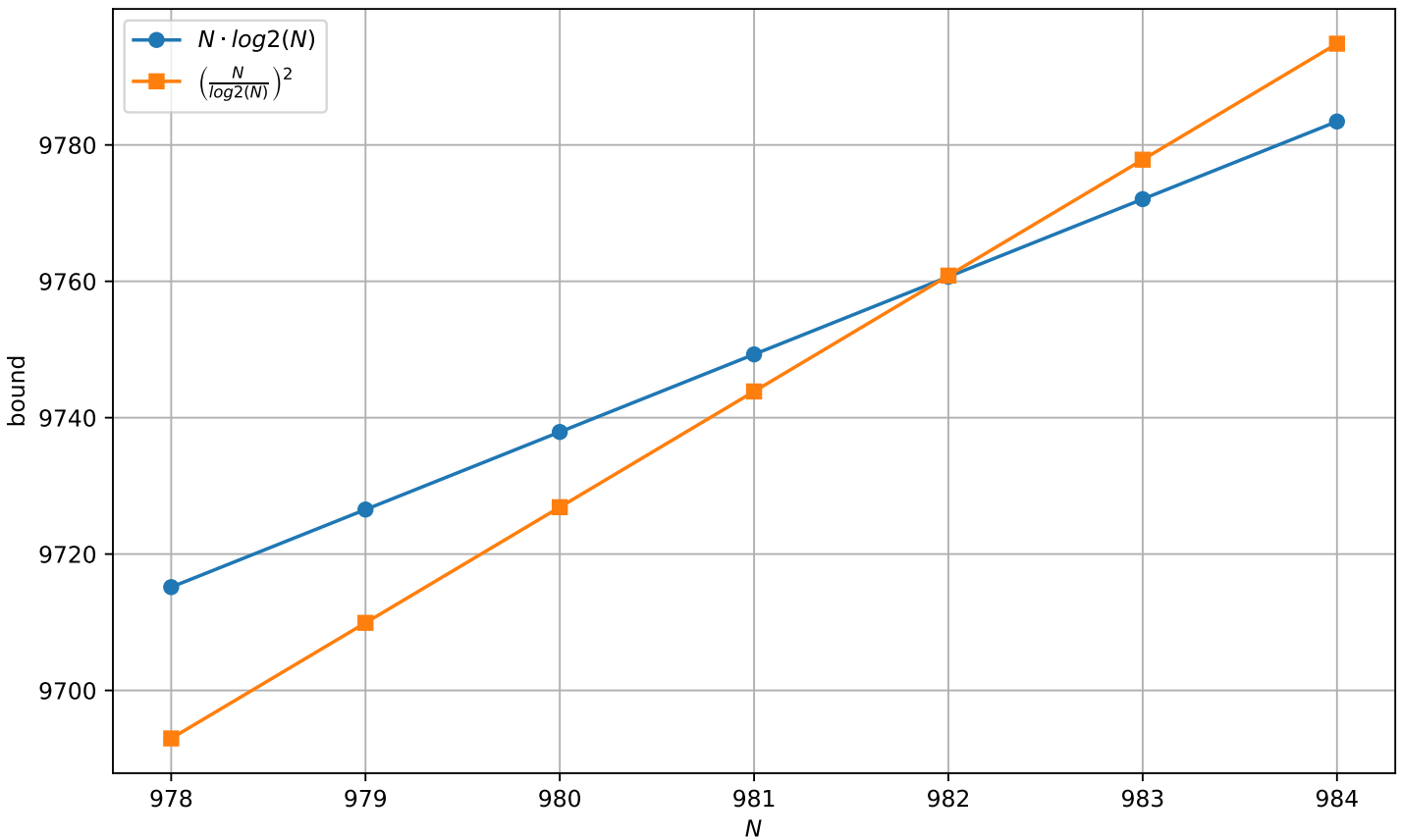
Communication complexity helped us to improve the AT^2 bound from

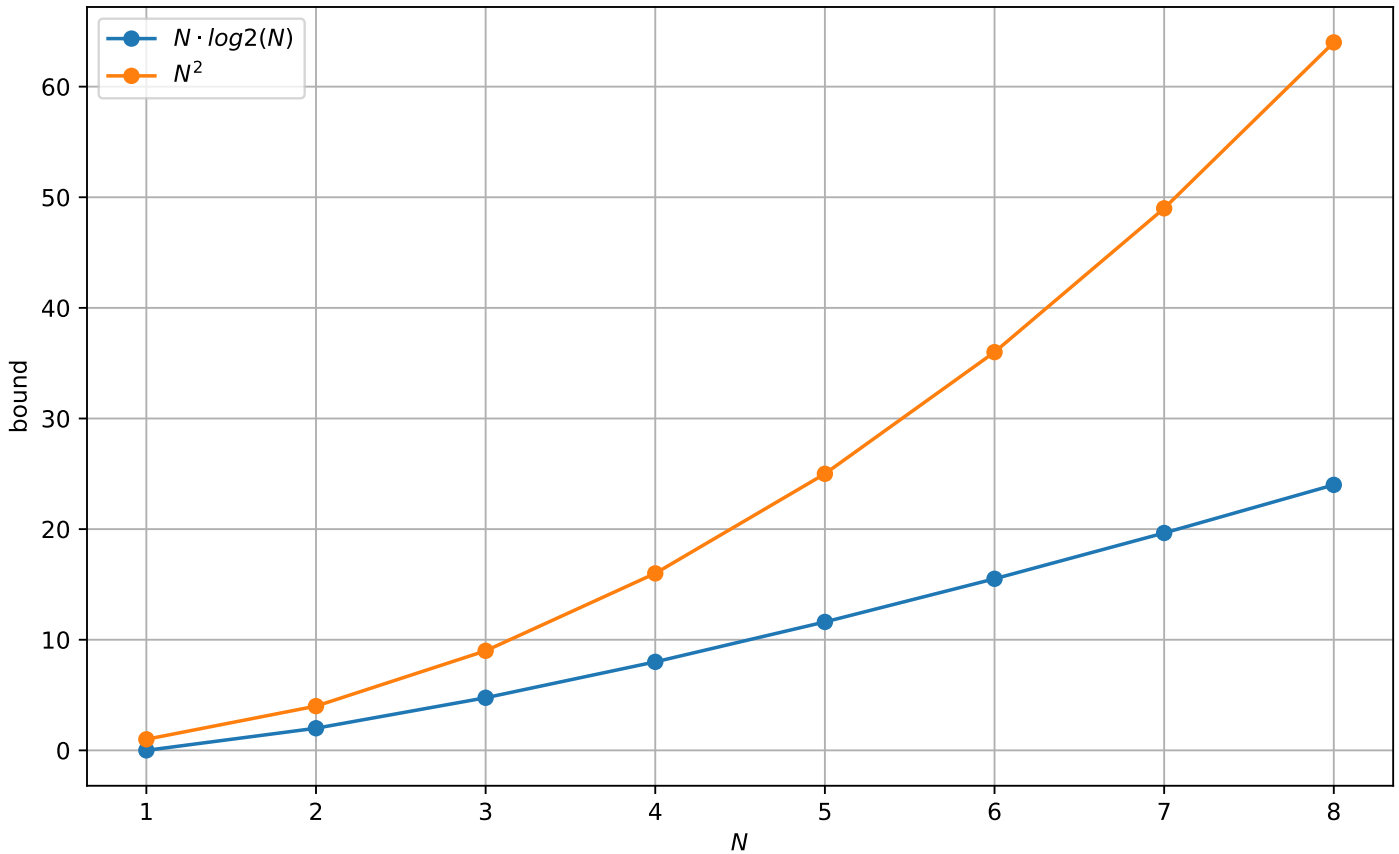
$$\Omega\left(N \cdot log_2(N)\right) \rightarrow \Omega\left(\frac{N^2}{log_2(N)^2}\right)$$



Bound from Communication Complexity

In most common practical cases, AT^2 improved from

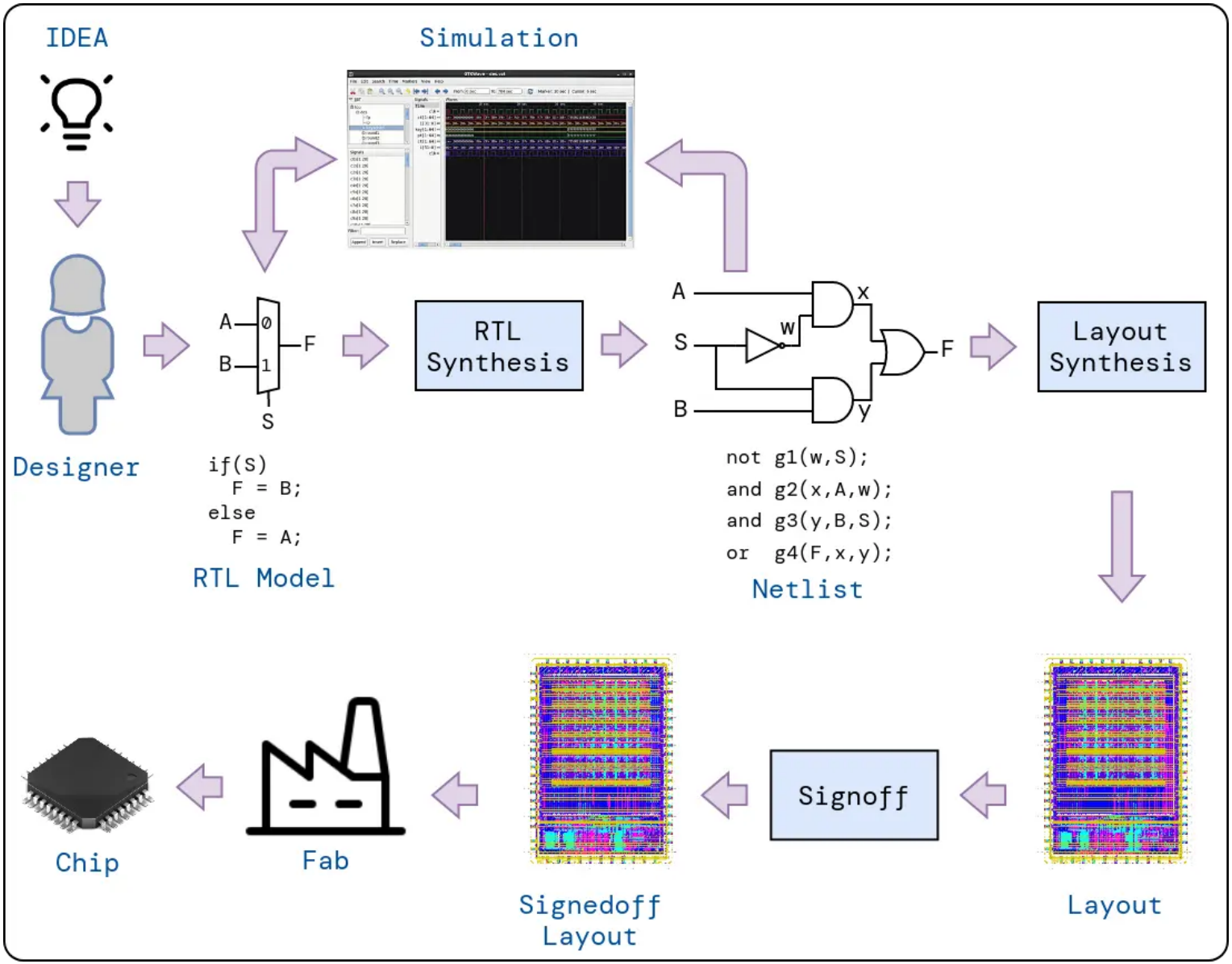
$$\Omega\left(N \cdot log_2(N)\right) \rightarrow \Omega\left(N^2\right)$$



Outline

Chip Design

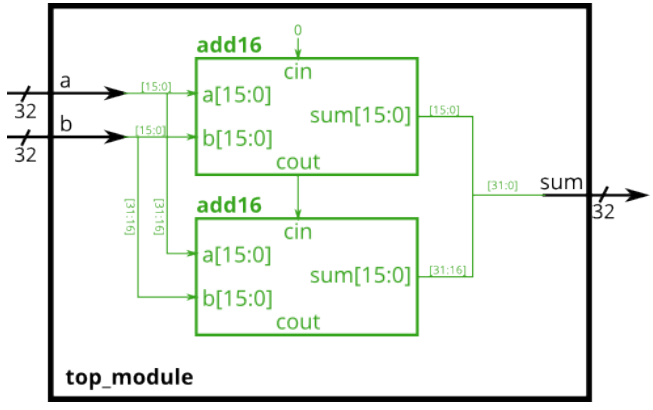
Overview



https://openlane2.readthedocs.io/en/latest/getting_started/newcomers/index.html

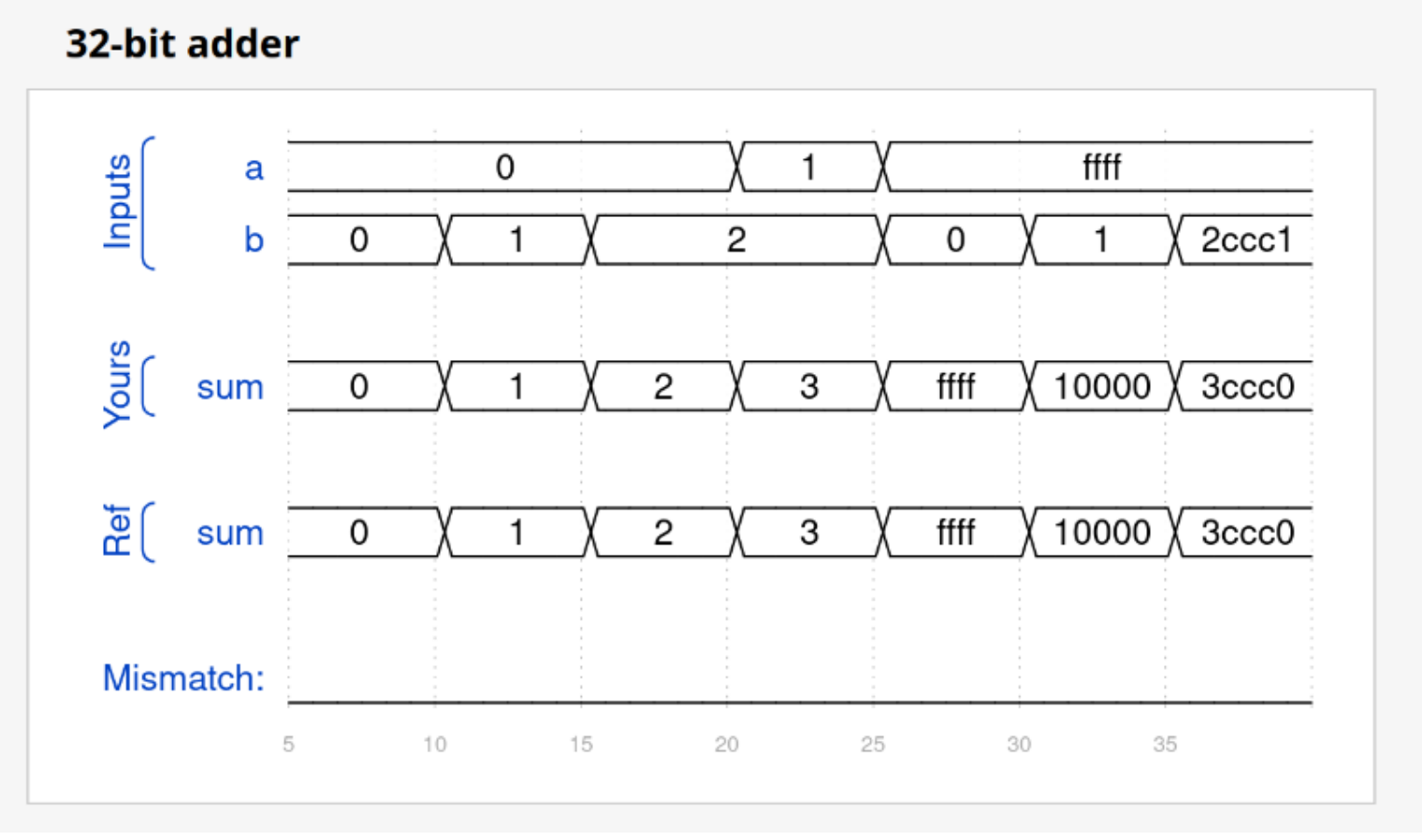
RTL Model: Verilog

```
1  module add32(
2      input [31:0] a,
3      input [31:0] b,
4      output [31:0] sum
5  );
6      wire carry;
7      wire [15:0]  sum_lo, sum_hi;
8
9      add16 a0 (a[15:0], b[15:0],  1'b0, sum_lo, carry);
10     add16 a1 (a[31:16], b[31:16], carry, sum_hi, 1'b0);
11
12     assign sum = {sum_hi, sum_lo};
13
14 endmodule
15
```



hdlbits.01xz.net

RTL Model: Test Bench (Simulation)



https://hdlbits.01xz.net/ using INTEL's Quartus

Process Design Kit (PDK)

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Defines Available gates, design rules, number of layers, etc.

Process Design Kit (PDK)

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Defines Available gates, design rules, number of layers, etc.

Feature	**Sky130**	**IHP SG13G2**
Technology Node	130nm	130nm
Designer	SkyWater Technology	Leibniz Institute for High Performance Microelectronics

Process Design Kit (PDK)

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Defines Available gates, design rules, number of layers, etc.

Feature	**Sky130**	**IHP SG13G2**
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Standard Cells	428	79

Process Design Kit (PDK)

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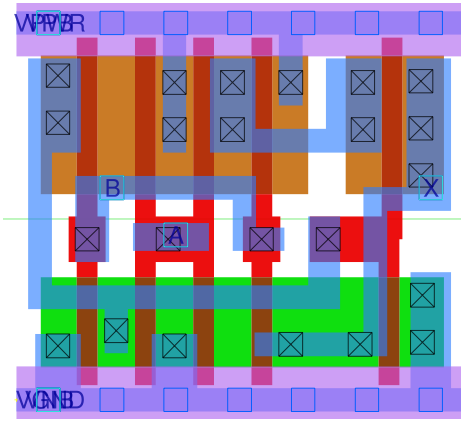
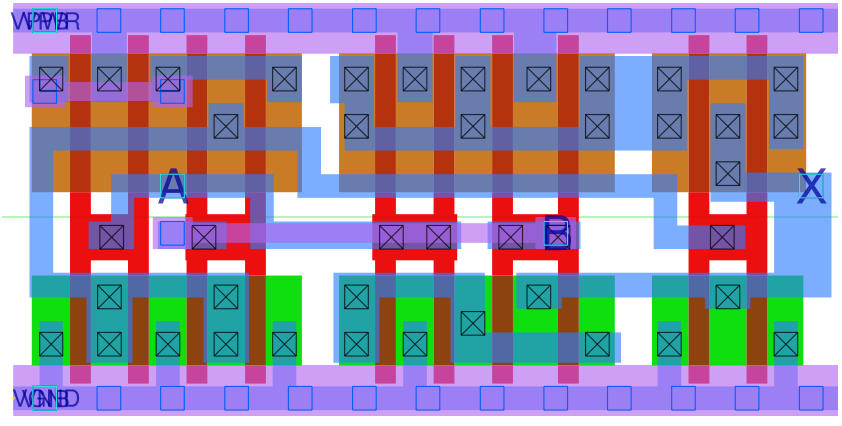
Defines Available gates, design rules, number of layers, etc.

Feature	**Sky130**	**IHP SG13G2**
Technology Node	130nm	130nm
Designer	SkyWater Technology	Leibniz Institute for High Performance Microelectronics
Standard Cells	428	79
Metal Layers	5	5–6

e.g. xor cell in `sky130_fd_sc_hd`



`xor2_1`



`xor2_2`

Same logic but with different drive strength.

RTL Synthesis (Netlist): Yosys

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RTL Synthesis (Netlist): Yosys

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```
        === expmob2 ===
        IHP PDK (Modified)
Number of cells:     599
   and21nor_x0         1
   and2_x1            35
   and3_x1            25
   and4_x1            18
   dff_x1             97
   inv_x0            114
   mux2_x1            1
   nand2_x0           49
   nand3_x0           6
   nand4_x0           5
   nexor2_x0          55
   ...
   nor3_x0            63
```

```
        === expmob2 ===
        Sky130 PDK
Number of cells:    388
  a221o_2           63
  a31o_2             3
  and2_2             8
  and2b_2           64
  and3_2            12
  and4_2            19
  buf_2             32
  or2_2              1
  or4_2              8
  or4bb_2            1
  ...
  xnor2_2            3
  xor2_2            40
```

RTL Synthesis (Netlist): Yosys

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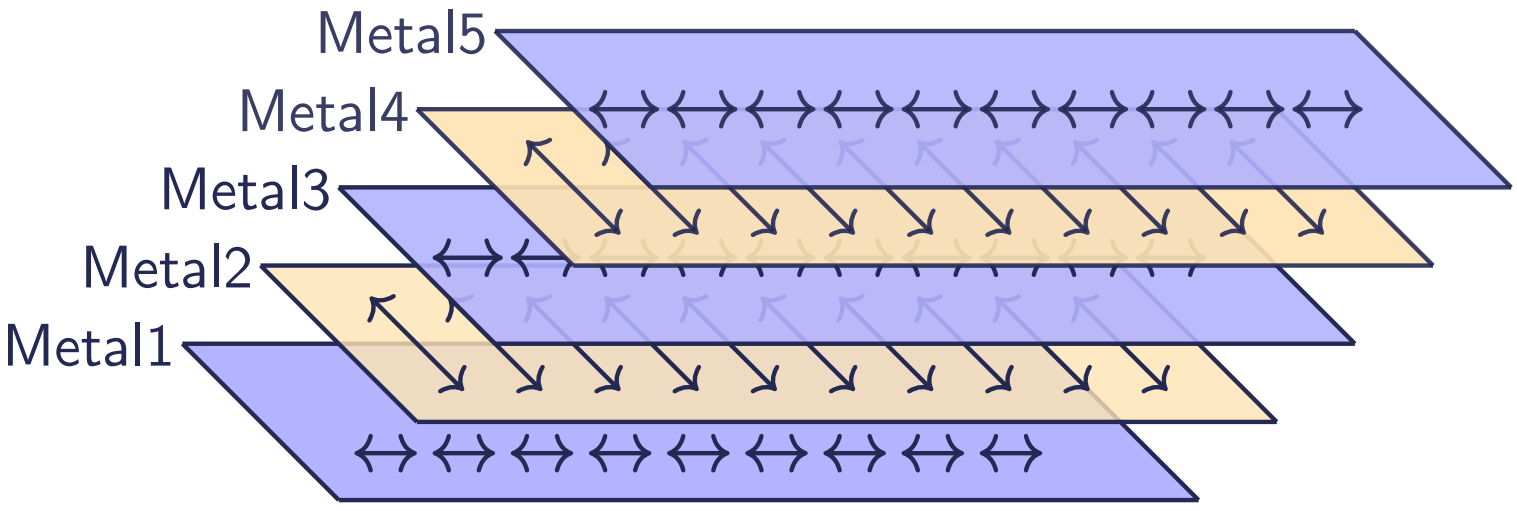
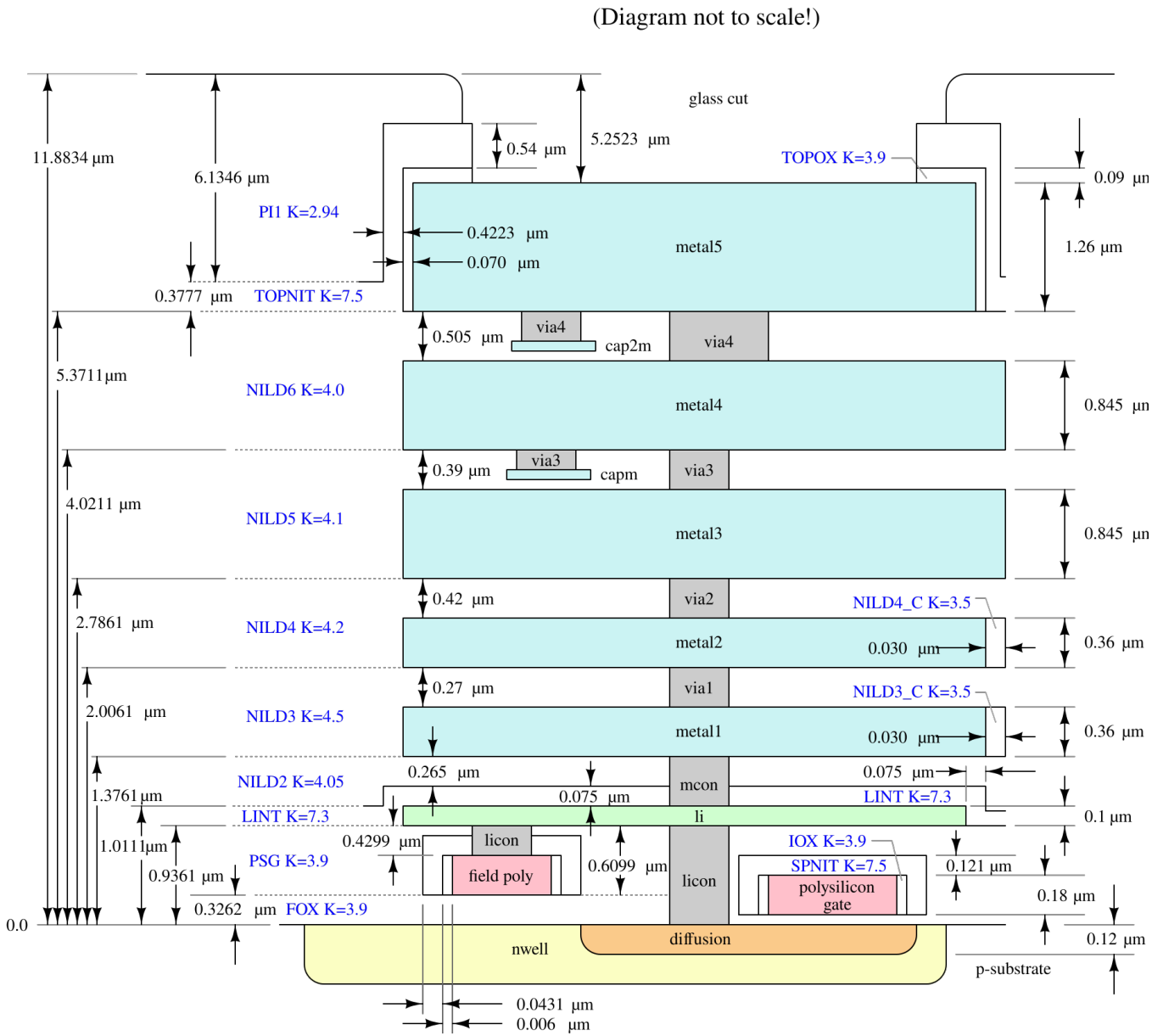
```
1  /* Generated by Yosys 0.46 (git sha1 e97731b9dda91fa5fa53ed87df7c34163ba59a41,
2  module expmob1(inputs, outputs);
3    wire _00_;
4     ...
5    wire _17_;
6    input [0:15] inputs;
7    wire [0:15] inputs;
8    output [0:15] outputs;
9    wire [0:15] outputs;
10   sky130_fd_sc_hd__xnor2_2 _18_ (
11     .A(inputs[8]),
12     .B(inputs[0]),
13     .Y(_00_)
14   );
15   sky130_fd_sc_hd__inv_2 _19_ (
16     .A(_00_),
17     .Y(outputs[8])
18   );
```

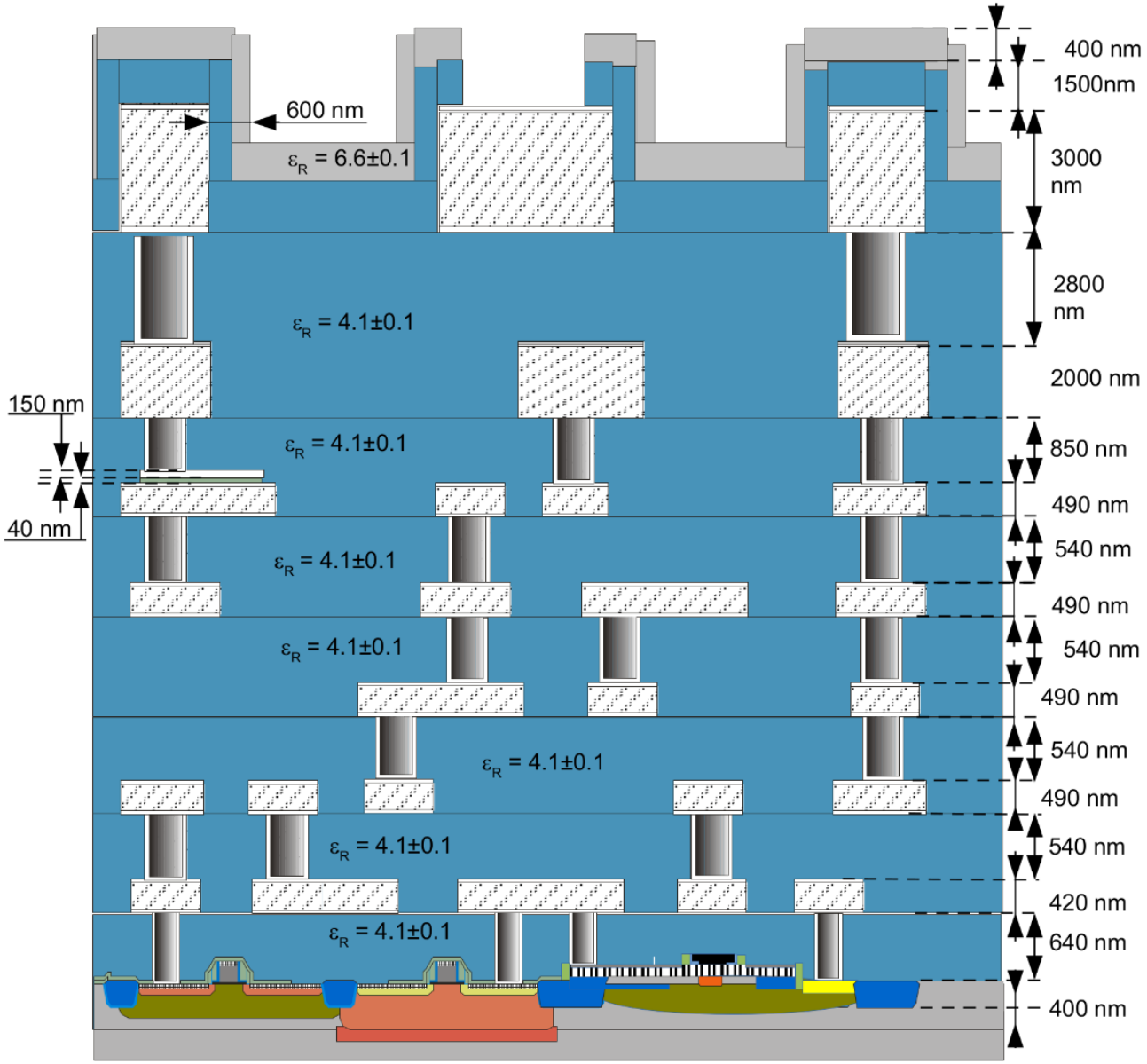
Routing Direction: e.g. Sky130



sky130: Cross section



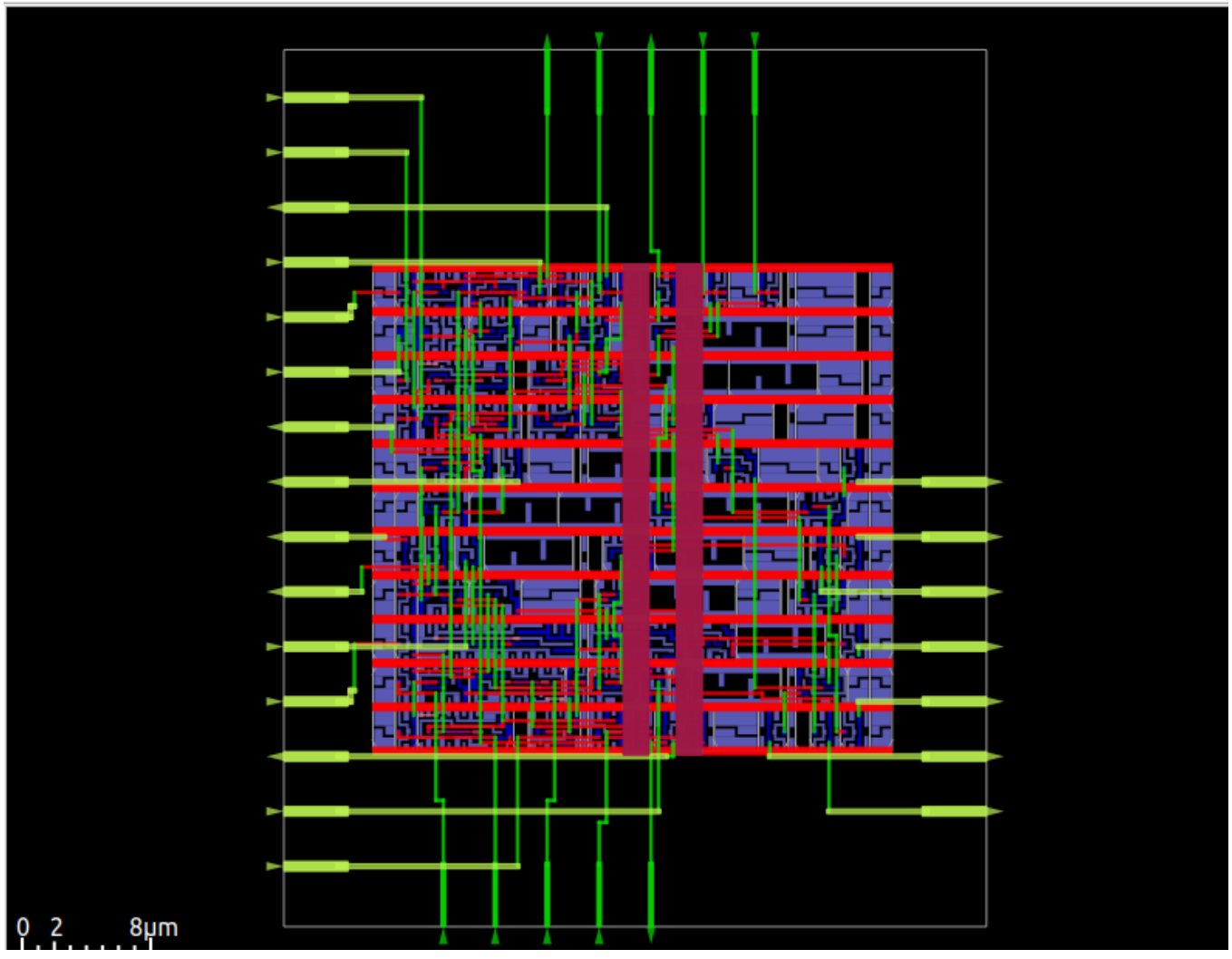
IHP: Cross section



More rules

- minimum wire width
- minimum distance between two wires (pitch)
- Components have to be on grid of 0.005 μm

Place and Route (PnR) or Layout Synthesis



`expmob1_n4 16 bits`

Comparison

Counting gates `expmob2_n14`

- Runtime `4m 44s`

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Counting gates `expmob2_n14`

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- Closer to circuit complexity

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PnR Layout `expmob2_n14`

- Runtime `19h 32m 37s`

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Counting gates `expmob2_n14`

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PnR Layout `expmob2_n14`

- Runtime `19h 32m 37s`

Gates Area	Wiring Area	Core Area
1222 μm^2	328μm^2	

`expmob2_n5`

Comparison

Counting gates `expmob2_n14`

- Runtime `4m 44s`
- Closer to circuit complexity

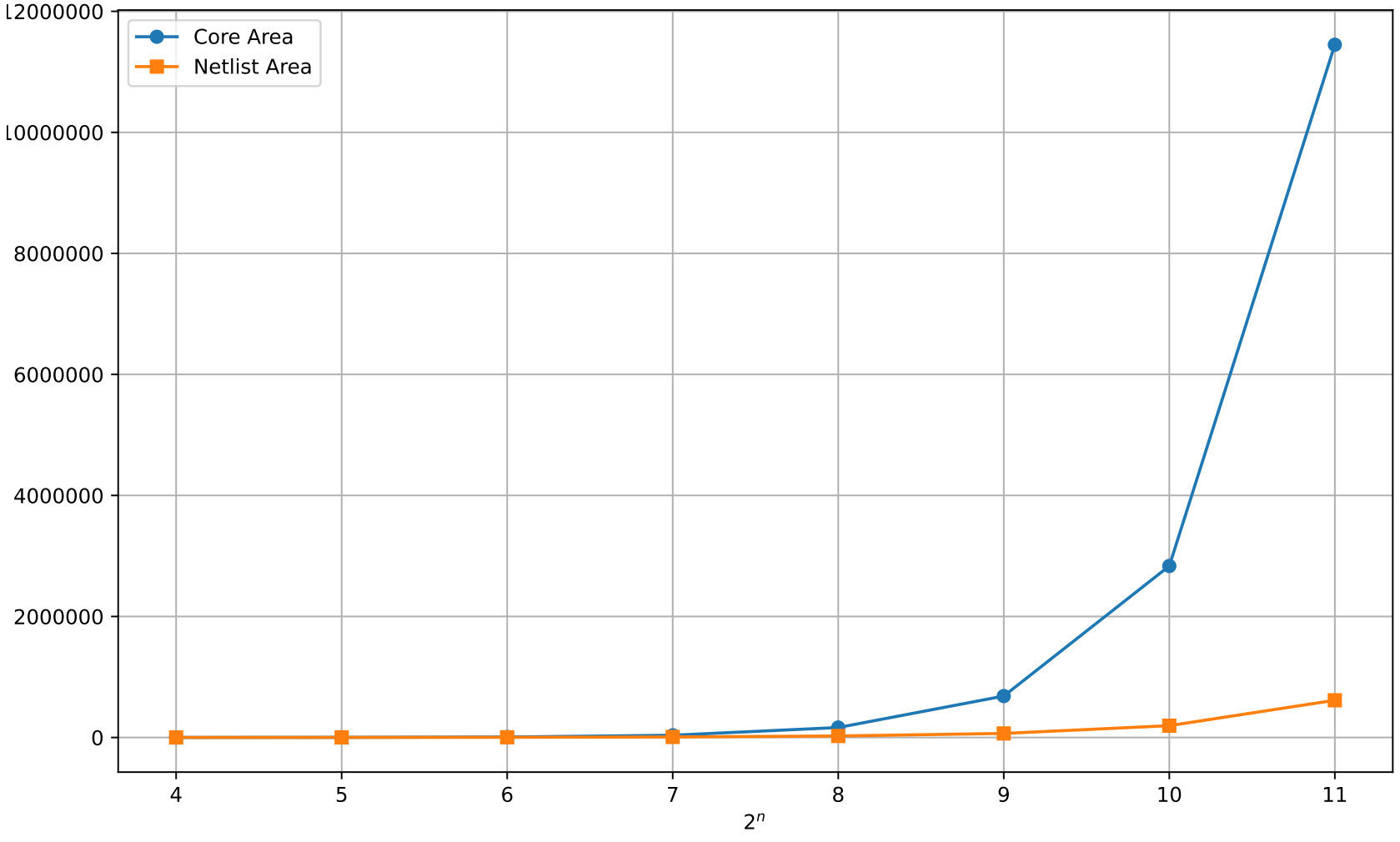
PnR Layout `expmob2_n14`

- Runtime `19h 32m 37s`

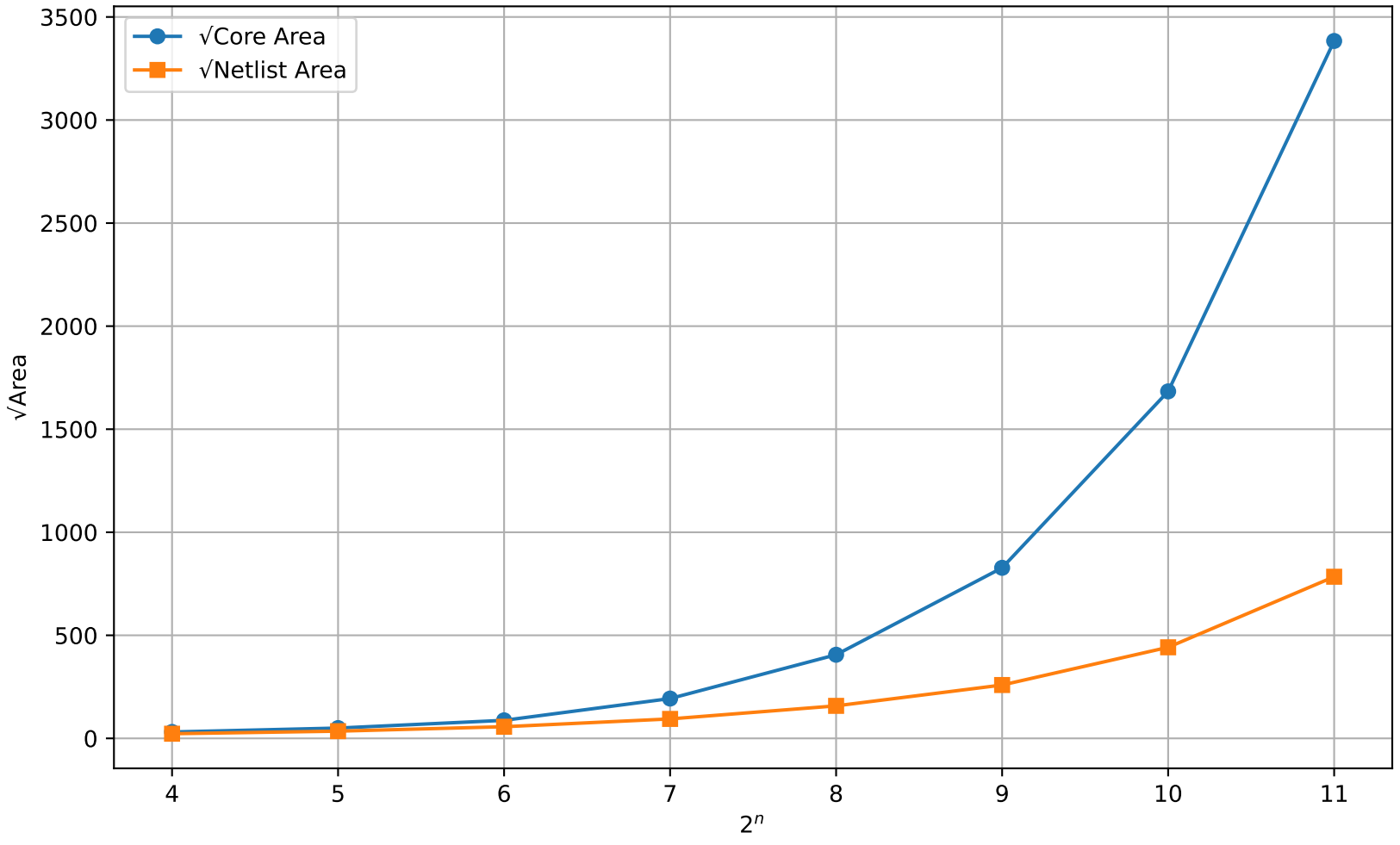
Gates Area	Wiring Area	Core Area
1222 μm^2	328μm^2	2499μm^2

`expmob2_n5`

Netlist Area vs. Layout area



Netlist Area vs. Layout area



Communication Complexity

Two-way Communication Complexity

$$f : X_A \times X_B \to Y_A \times Y_B, \quad |X_A| = |X_B|$$

Alice	**Bob**
Knows X_A	Knows X_B
Wishes to compute Y_A	Wishes to know Y_B
$\xrightarrow{m_0}$	
$\xleftarrow{m_1}$	
$\dots$	
computes y_a	computes y_b

Two-way Communication Complexity

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$$\xrightarrow{\quad m_0 \quad}$$

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$$\dots$$

computes y_a computes y_b

Communication complexity := minimum number of bits exchanged

Two-way Communication Complexity

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Alice **Bob**

Knows X_A Knows X_B

Wishes to compute Y_A Wishes to know Y_B

$m_0 \longrightarrow$

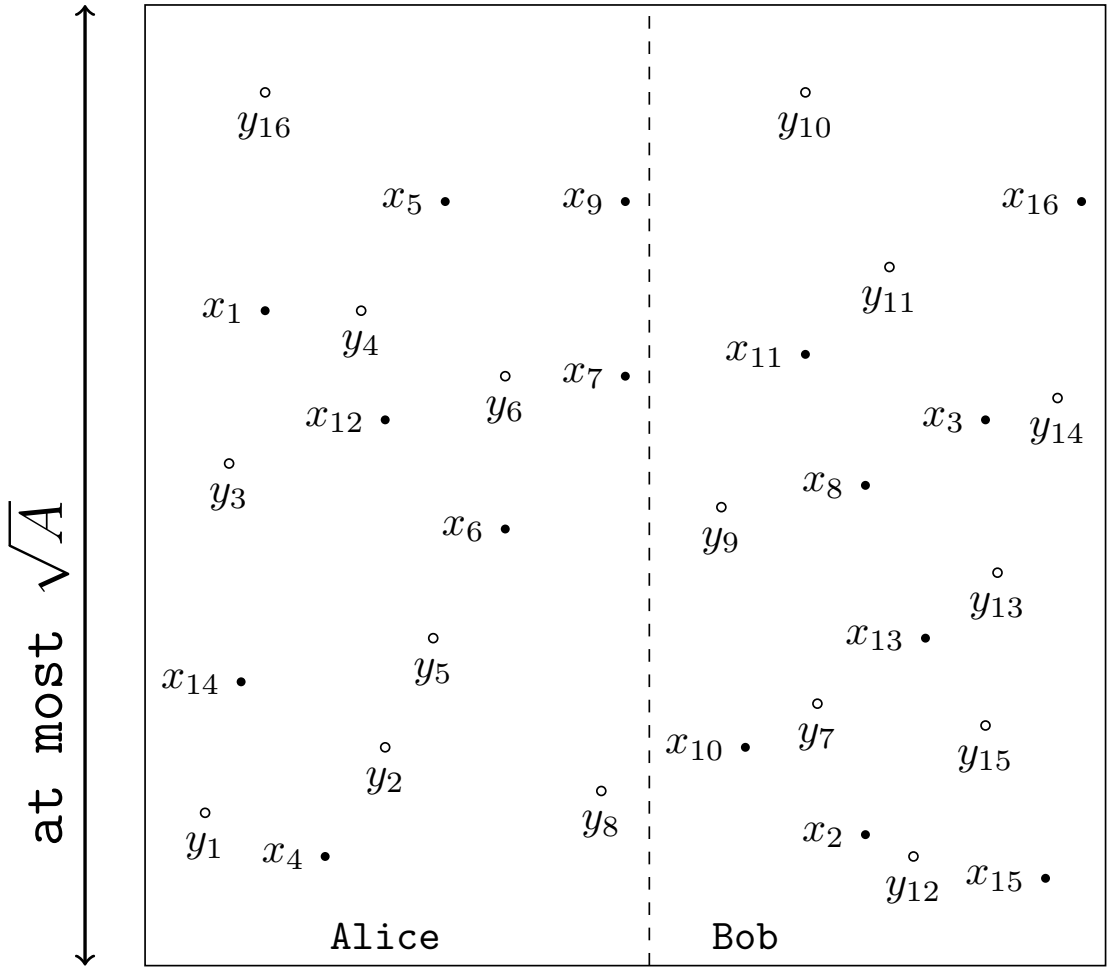
$\longleftarrow m_1$

$\ldots$

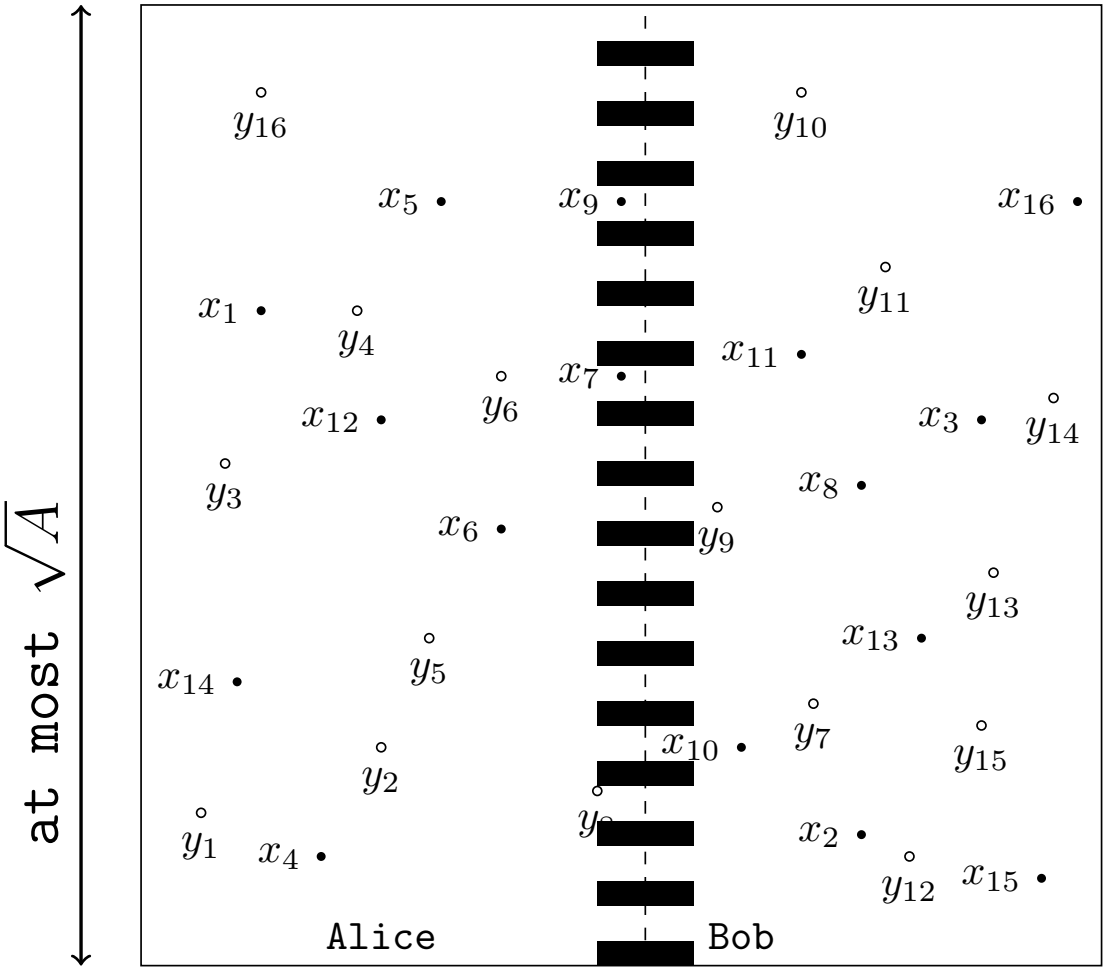
computes y_a computes y_b

Communication complexity := minimum number of bits exchanged
Both parties **don't** have to learn all the output bits.

Relationship to circuits: space separation

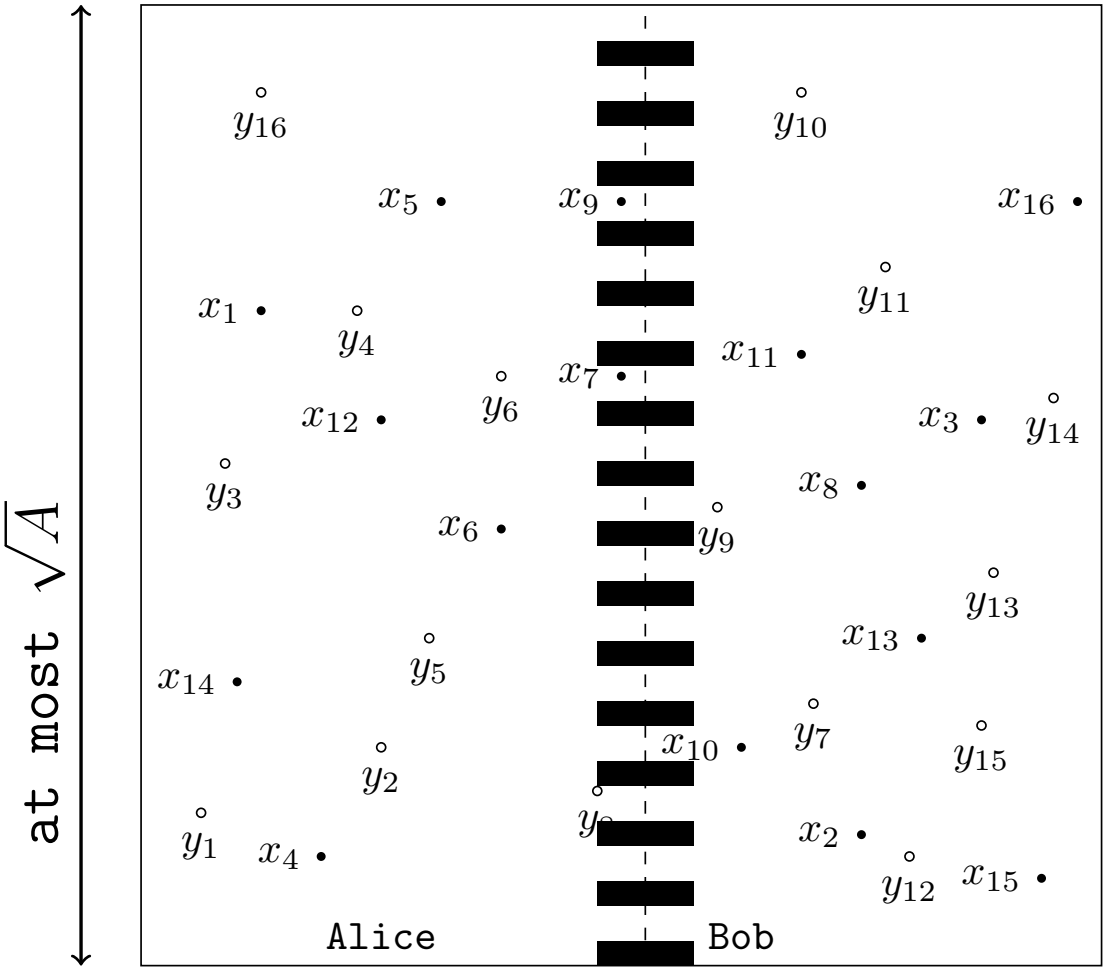


Relationship to circuits: space separation



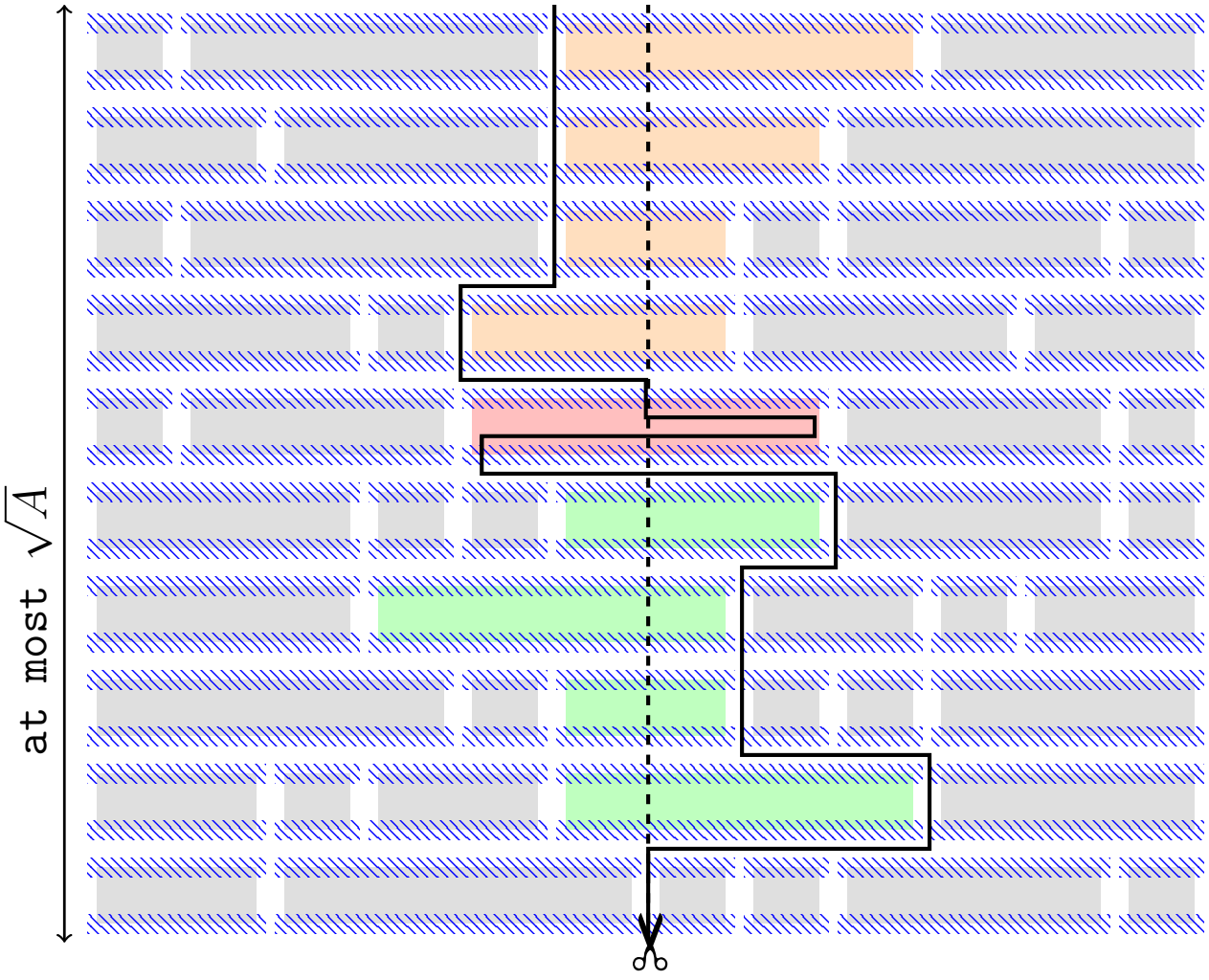
1. At most $\Omega\left(\sqrt{A}\right)$ wires cross both sections

Relationship to circuits: space separation



1. At most $\Omega\left(\sqrt{A}\right)$ wires cross both sections
2. After `T` `cycles`, at most $\Omega\left(\sqrt{A} \cdot T\right)$ bits are exchanged

Relationship to circuits: space separation PDK



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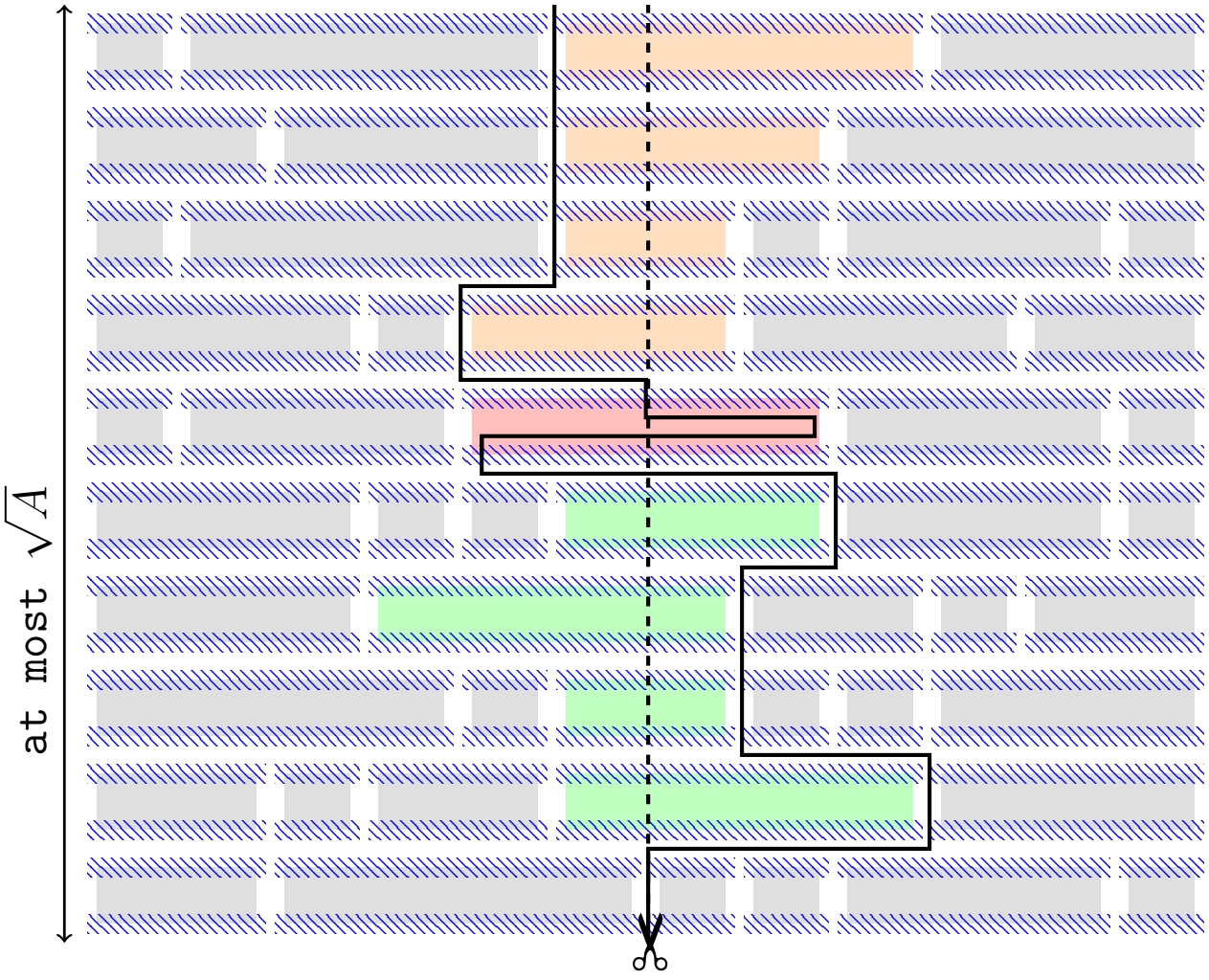
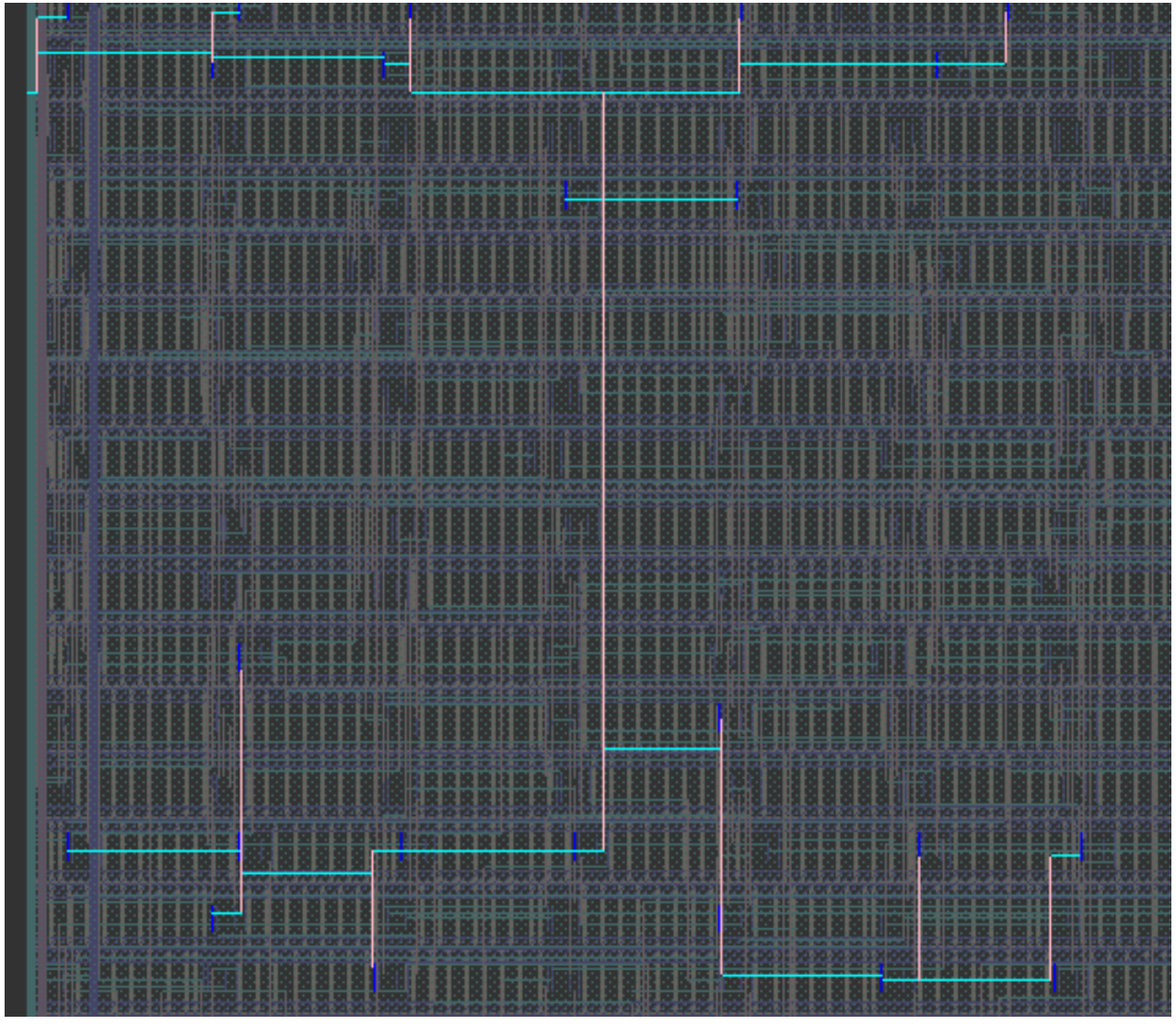


$$\sqrt{A} \cdot T \geq \Omega\left(twCC(f)\right)$$

e.g. Rotation circuit



Wires that deliver a single input

Time separation

If all input ports all present then $A \geq \Omega(n)$

Time separation

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Assume, the function has `64-bits` inputs, but the circuit accepts only `8-bits` of the input at a time unit.

Time separation

If all input ports all present then $A \geq \Omega(n)$

Assume, the function has `64-bits` inputs, but the circuit accepts only `8-bits` of the input at a time unit.

How can we bound the area A?

One-Way Communication Complexity

How to compute $f : \{0,1\}^{2n} \to \{0,1\}^{2m}$?

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Alice **Bob**

Knows X_A Knows X_B

b_0 ⟶

b_1 ⟶

$\dots$

computes y_b

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Alice **Bob**

Knows X_A Knows X_B

b_0 ⟶

b_1 ⟶

$\dots$

computes y_b

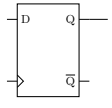
Communication complexity := minimum number of bits Alice sends

Relationship to circuits: time separation

Alice **Bob**

Simulate the circuit

for first 1/2 inputs



memory contents

$\longrightarrow$

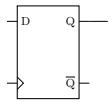
computes y_b

Relationship to circuits: time separation

Alice **Bob**

Simulate the circuit

for first 1/2 inputs



memory contents

$\longrightarrow$

computes y_b

$$A \geq \Omega\left(owCC(f)\right)$$

Bounding techniques

Proof techniques: Flow argument

$$f : X_A \times X_B \to Y_A \times Y_B, |X_A| = |X_B|$$

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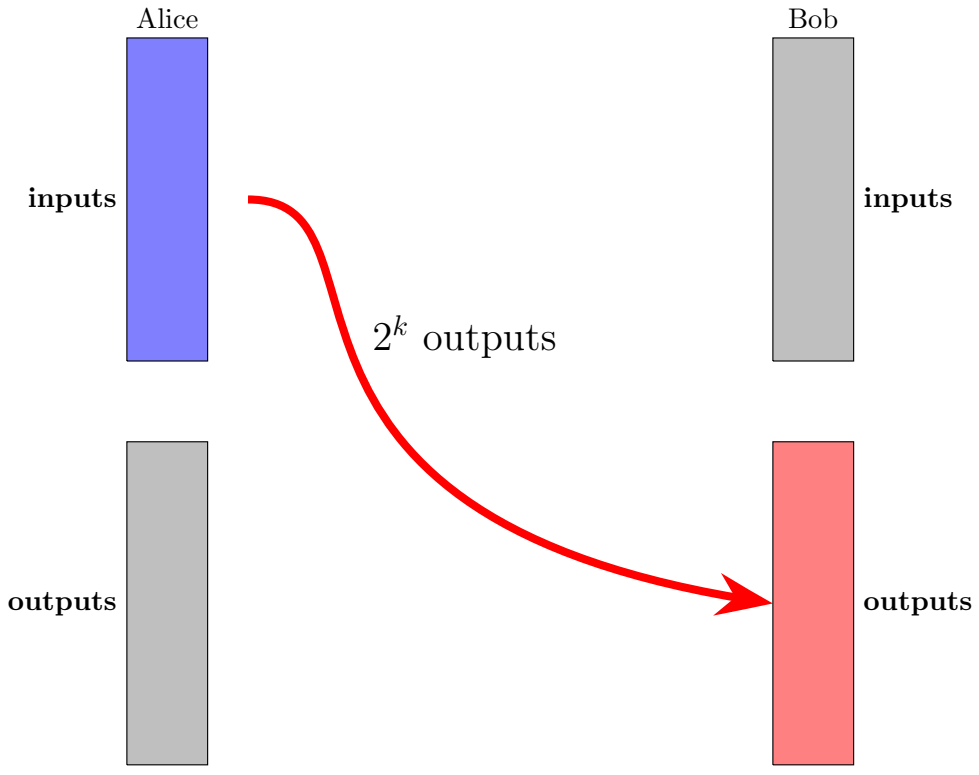

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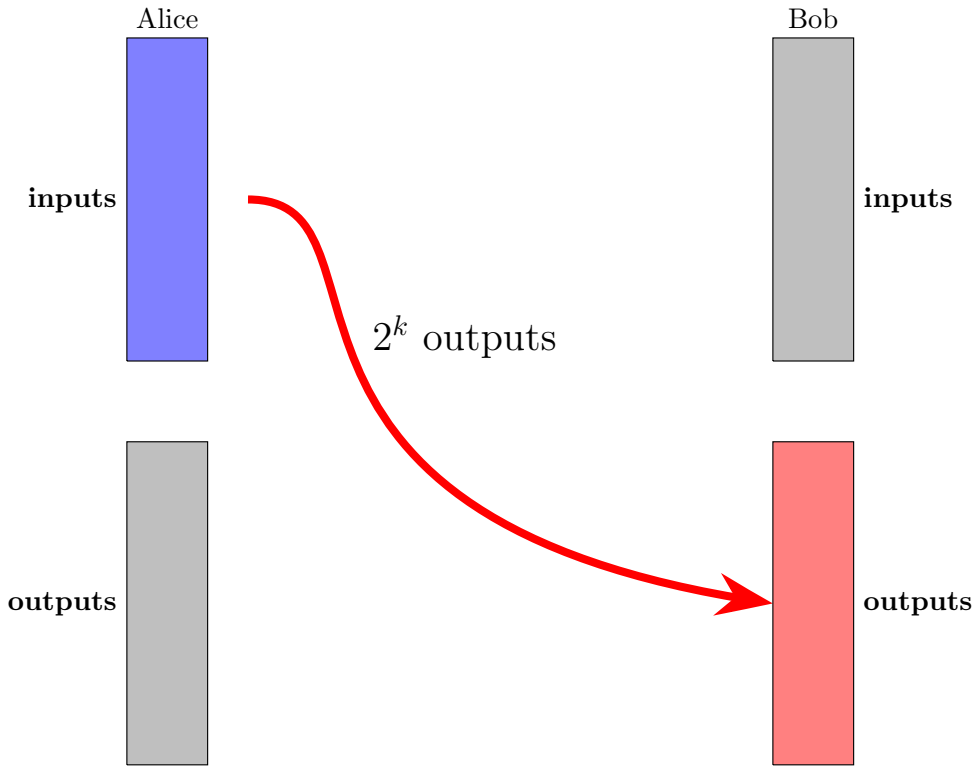
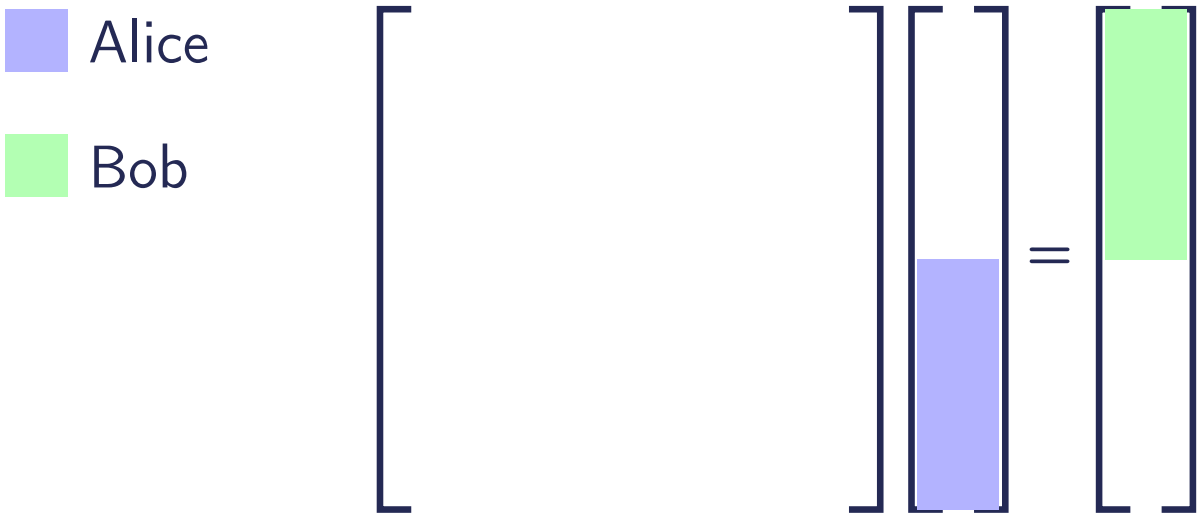


If Alice can generate 2^k different outputs in Bob's part, then she must send at least k-bits.

Proof techniques: Rank method $f(x) := Ax = b$



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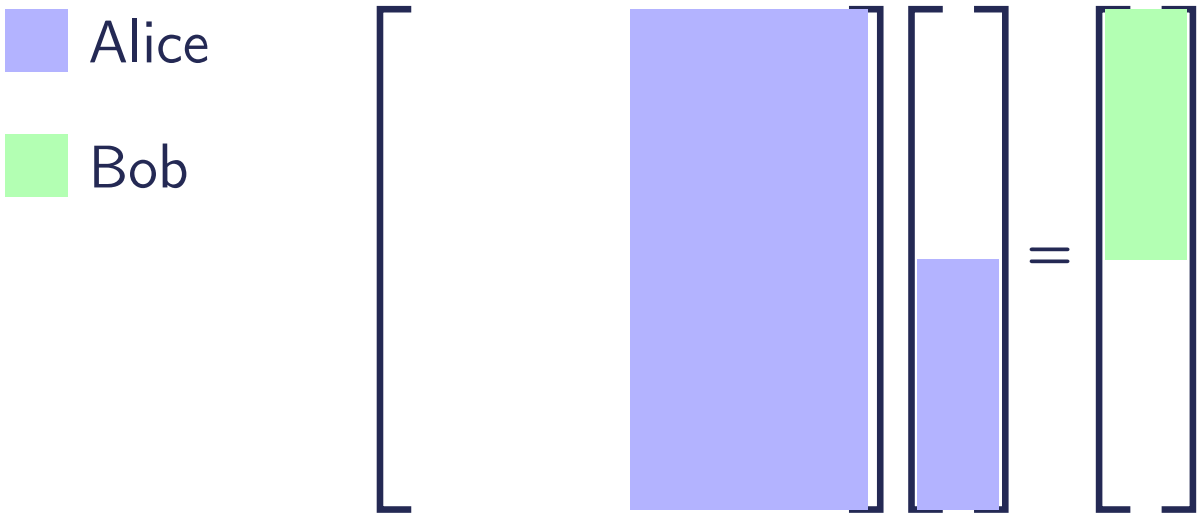

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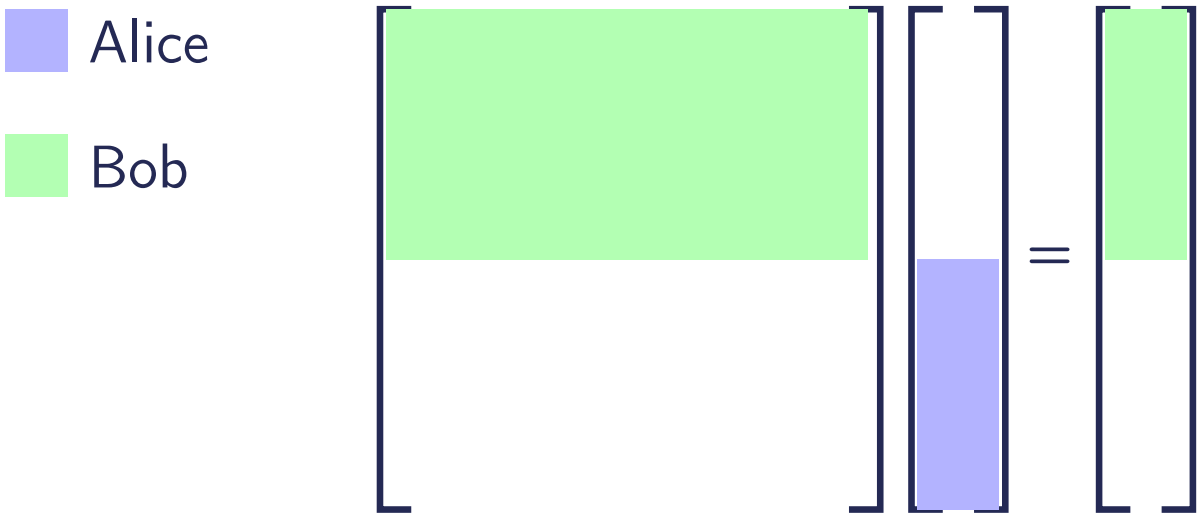

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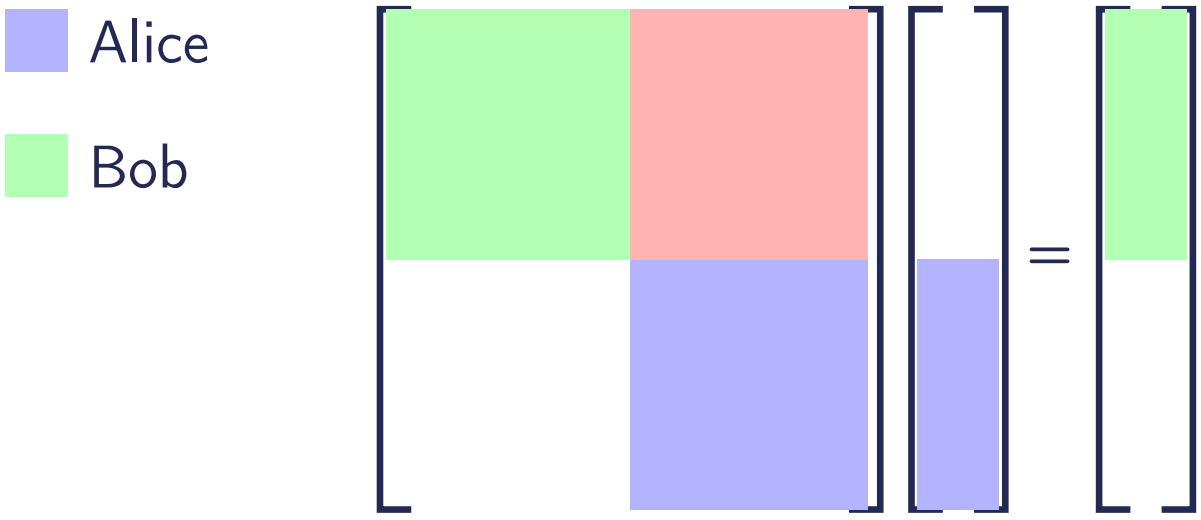

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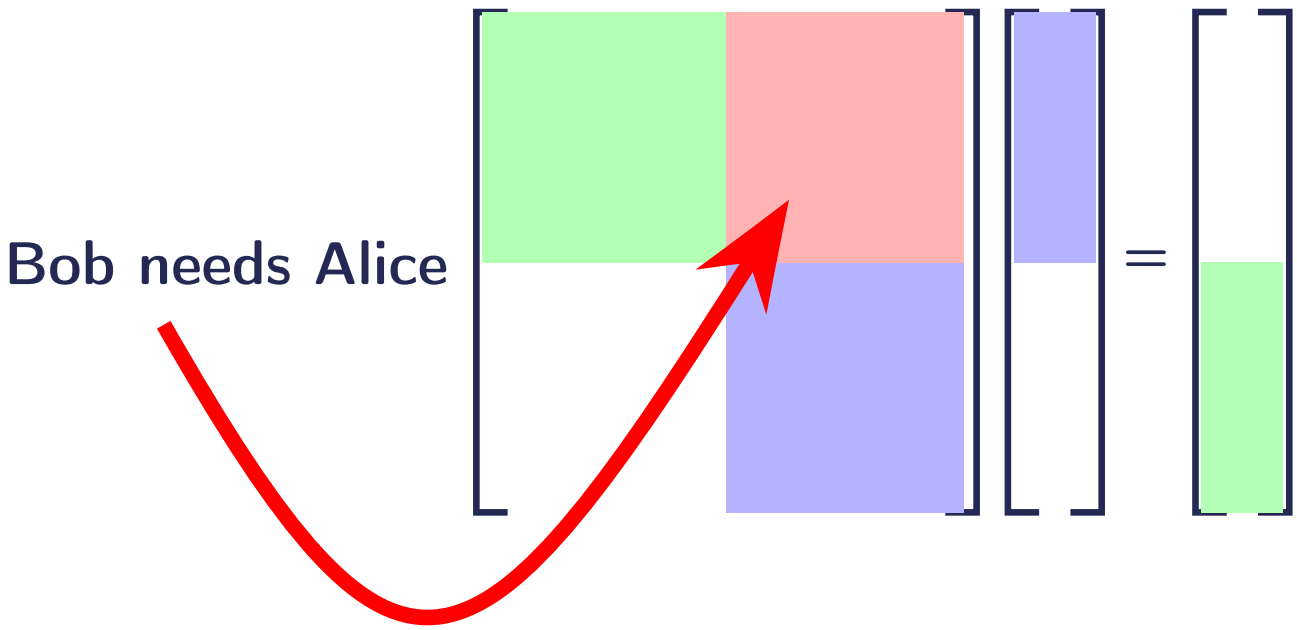

Reduce it to a binary problem

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Numerous techniques: *Rectangles, fooling sets, rank, etc...*

Möbius Transform

Definition

Möbius Transform

It converts truth table of a Boolean function to ANF, and vice versa.

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- $M_n = \begin{pmatrix} M_{n-1} & 0 \\ M_{n-1} & M_{n-1} \end{pmatrix}$

From now on, we assume there are $2N$ i.e. Möbius_{2N}

e.g. N=32

$$
\left(\begin{array}{cccccccc|cccccccc||cccccccc|cccccccc}
1 & & & & & & & & 1 & & & & & & & & & & & & & & & & & & & & \\
1 & 1 & & & & & & & 1 & 1 & & & & & & & & & & & & & & & & & & & \\
1 & & 1 & & & & & & 1 & & 1 & & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & & & & & 1 & 1 & 1 & 1 & & & & & & & & & & & & & & & & \\
1 & & & & 1 & & & & 1 & & & & 1 & & & & & & & & & & & & & & & & \\
1 & 1 & & & 1 & 1 & & & 1 & 1 & & & 1 & 1 & & & & & & & & & & & & & & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & & & & & & & \\
\hline
1 & & & & & & & & 1 & & & & & & & & 1 & & & & & & & & & & & & \\
1 & 1 & & & & & & & 1 & 1 & & & & & & & 1 & 1 & & & & & & & & & & & \\
1 & & 1 & & & & & & 1 & & 1 & & & & & & 1 & & 1 & & & & & & & & & & \\
1 & 1 & 1 & 1 & & & & & 1 & 1 & 1 & 1 & & & & & 1 & 1 & 1 & 1 & & & & & & & & & \\
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1 & 1 & & & 1 & 1 & & & 1 & 1 & & & 1 & 1 & & & 1 & 1 & & & 1 & 1 & & & & & & & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
\hline\hline
1 & & & & & & & & & & & & & & & & 1 & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & & & & & 1 & 1 & & & & & & & & & & & \\
1 & & 1 & & & & & & & & & & & & & & 1 & & 1 & & & & & & & & & & \\
1 & 1 & 1 & 1 & & & & & & & & & & & & & 1 & 1 & 1 & 1 & & & & & & & & & \\
1 & & & & 1 & & & & & & & & & & & & 1 & & & & 1 & & & & & & & & \\
1 & 1 & & & 1 & 1 & & & & & & & & & & & 1 & 1 & & & 1 & 1 & & & & & & & \\
1 & & 1 & & 1 & & 1 & & & & & & & & & & 1 & & 1 & & 1 & & 1 & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
\hline
1 & & & & & & & & 1 & & & & & & & & 1 & & & & & & & & 1 & & & & \\
1 & 1 & & & & & & & 1 & 1 & & & & & & & 1 & 1 & & & & & & & 1 & 1 & & & \\
1 & & 1 & & & & & & 1 & & 1 & & & & & & 1 & & 1 & & & & & & 1 & & 1 & & \\
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1 & & & & 1 & & & & 1 & & & & 1 & & & & 1 & & & & 1 & & & & 1 & & & \\
1 & 1 & & & 1 & 1 & & & 1 & 1 & & & 1 & 1 & & & 1 & 1 & & & 1 & 1 & & & 1 & 1 & & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)
\left(\begin{array}{c}
x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ \hline\hline
x_{16} \\ x_{17} \\ x_{18} \\ x_{19} \\ x_{20} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \\ x_{25} \\ x_{26} \\ x_{27} \\ x_{28} \\ x_{29} \\ x_{30} \\ x_{31}
\end{array}\right)
$$

As an equality tool

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$$
\begin{pmatrix}
1 \\
1&1 \\
1&&1 \\
1&1&1&1 \\
1&&&&1 \\
1&1&&&1&1 \\
1&&1&&1&&1 \\
1&1&1&1&1&1&1&1 \\
1&&&&&&&&1 \\
1&1&&&&&&&1&1 \\
1&&1&&&&&&1&&1 \\
1&1&1&1&&&&&1&1&1&1 \\
1&&&&1&&&&1&&&&1 \\
1&1&&&1&1&&&1&1&&&1&1 \\
1&&1&&1&&1&&1&&1&&1&&1 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\
1&&&&&&&&&&&&&&&&1 \\
1&1&&&&&&&&&&&&&&&1&1 \\
1&&1&&&&&&&&&&&&&&1&&1 \\
1&1&1&1&&&&&&&&&&&&&1&1&1&1 \\
1&&&&1&&&&&&&&&&&&1&&&&1 \\
1&1&&&1&1&&&&&&&&&&&1&1&&&1&1 \\
1&&1&&1&&1&&&&&&&&&&1&&1&&1&&1 \\
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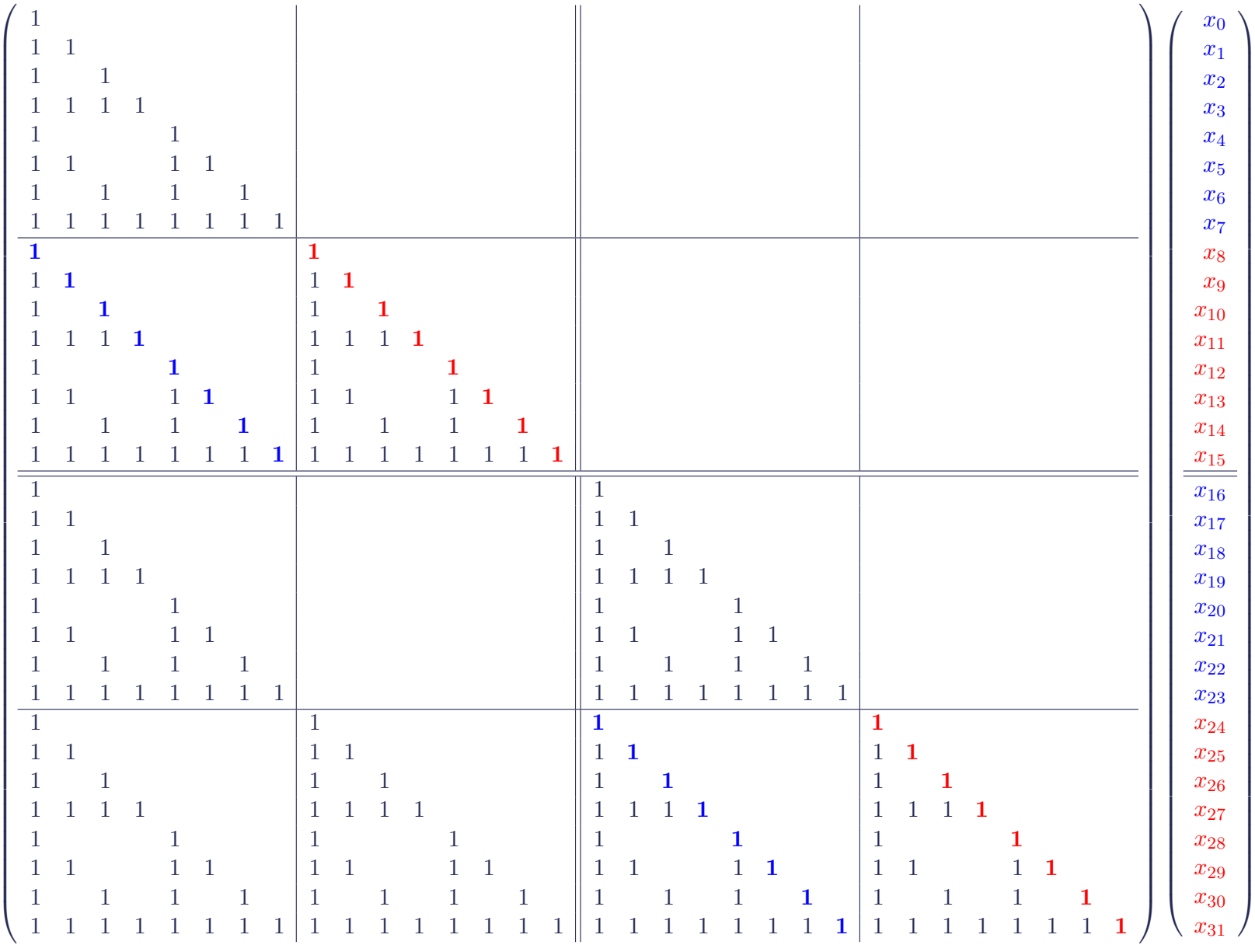
1&&&&&&&&1&&&&&&&&1&&&&&&&&1 \\
1&1&&&&&&&1&1&&&&&&&1&1&&&&&&&1&1 \\
1&&1&&&&&&1&&1&&&&&&1&&1&&&&&&1&&1 \\
1&1&1&1&&&&&1&1&1&1&&&&&1&1&1&1&&&&&1&1&1&1 \\
1&&&&1&&&&1&&&&1&&&&1&&&&1&&&&1&&&&1 \\
1&1&&&1&1&&&1&1&&&1&1&&&1&1&&&1&1&&&1&1&&&1&1 \\
1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1 \\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}
\begin{pmatrix}
x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \\ x_{18} \\ x_{19} \\ x_{20} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \\ x_{25} \\ x_{26} \\ x_{27} \\ x_{28} \\ x_{29} \\ x_{30} \\ x_{31}
\end{pmatrix}
$$

As an equality tool



Reflections: twCC for Möbius over 2N bits

- We know $twCC\left(\mathsf{eq}_{2N}\right) \geq N$

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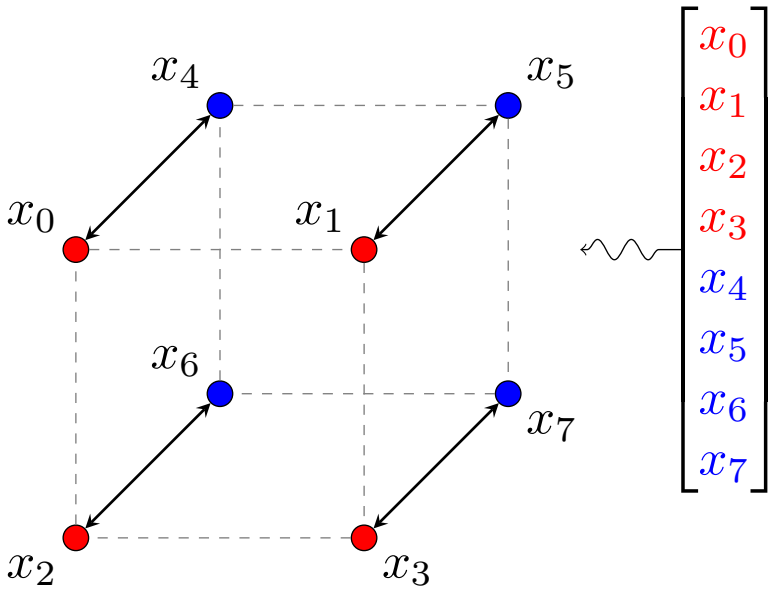
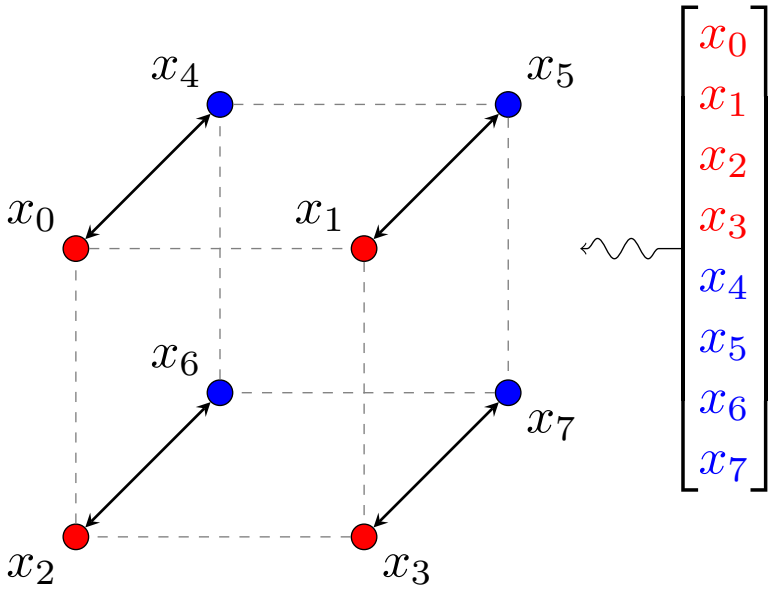
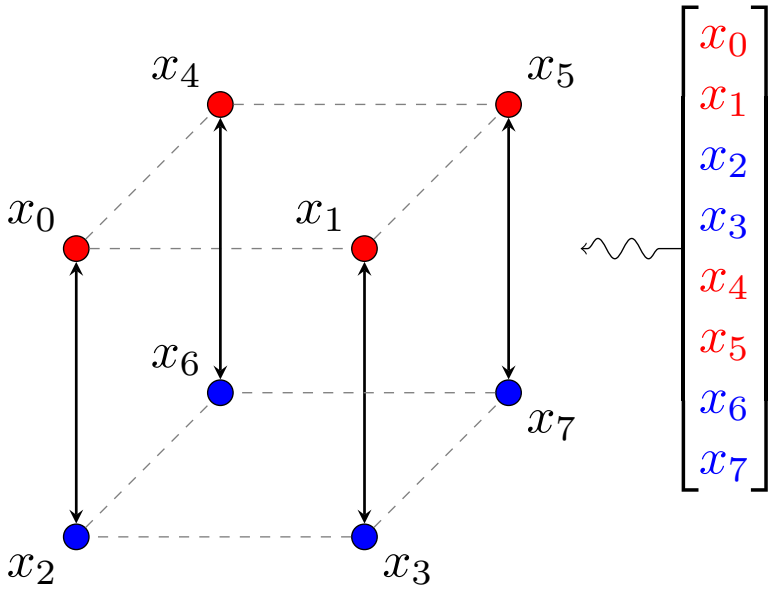
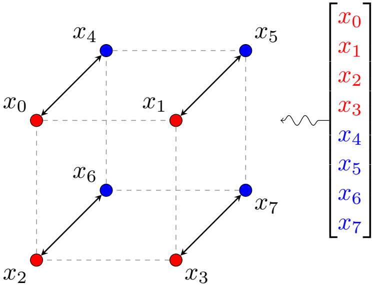
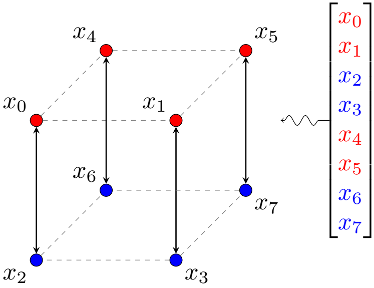
- Our reduction $\mathsf{M\ddot{o}bius}_{2N} \to \mathsf{eq}_{2N}$ works only on $2^N \cdot log_2\left(2N\right)$ partitions, but there are $\binom{2N}{N}$ partitions.

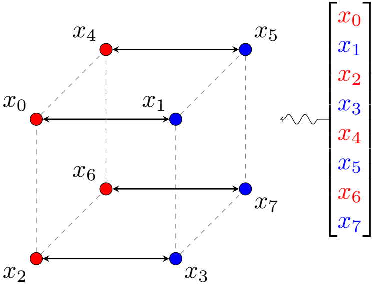

x_4
x_5
x_0
x_1
x_6
x_7
x_2
x_3
x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7

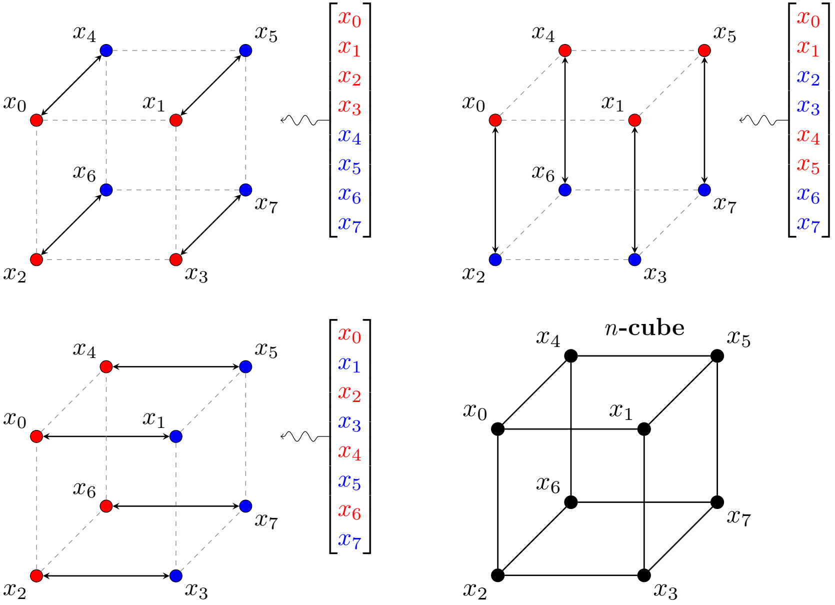

x_4
x_5
x_0
x_1
x_6
x_7
x_2
x_3
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7


x_4
x_5
x_0
x_1
x_6
x_7
x_2
x_3
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7


x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7

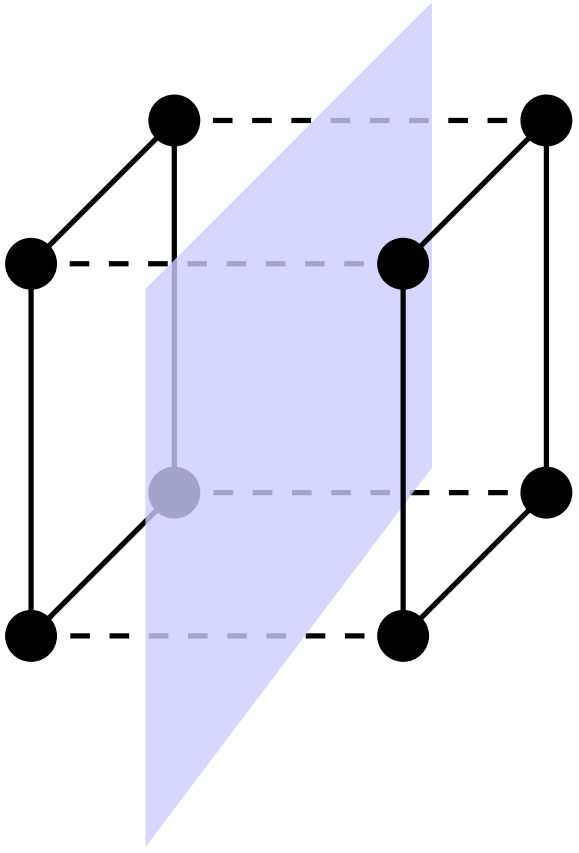

x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7


x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7


x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7
n-cube

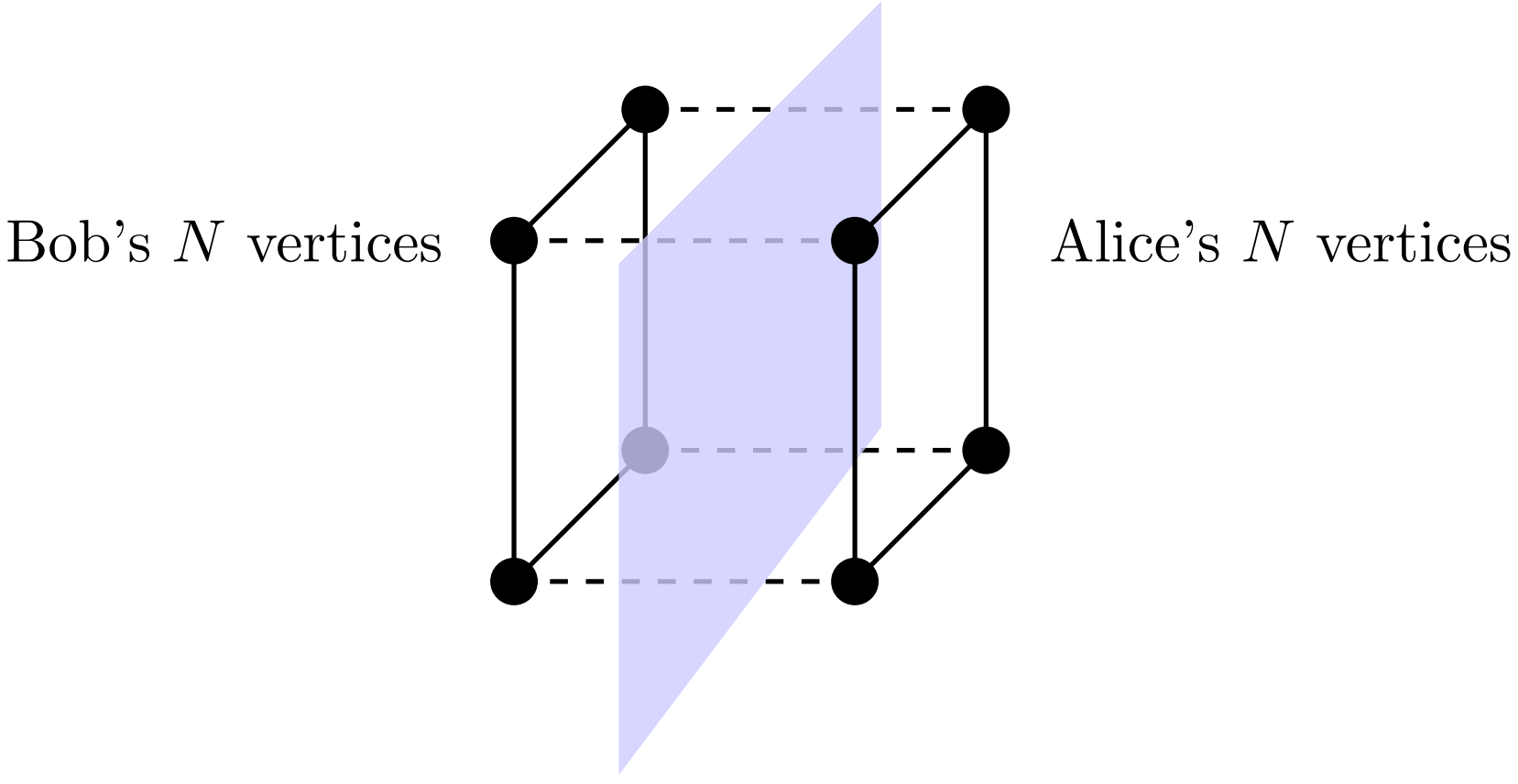
Isoperimetric Inequality: $n+1$-cube, $2N$ vertices

Bob's N vertices



Alice's N vertices

Isoperimetric Inequality: $n+1$-cube, $2N$ vertices



∂X_A is the set of edges between Alice and Bob, thus

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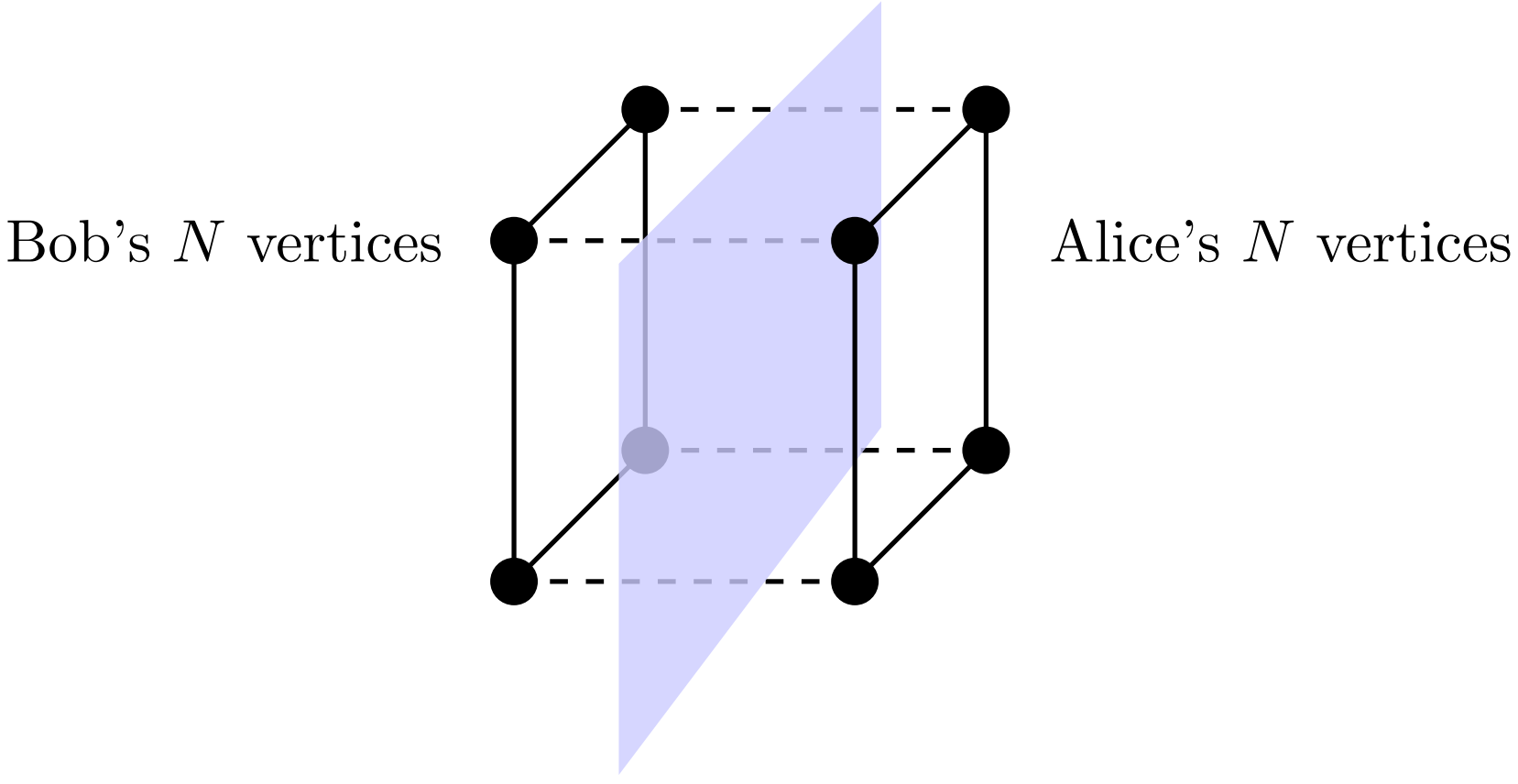

∂X_A is the set of edges between Alice and Bob, thus
$\partial X_A \geq log_2(\frac{2N}{N}) \cdot N = N$ edges between them

- $N \cdot log_2\,(2N)$ comparisons using the $log_2\,(2N)$ canonical partitions.

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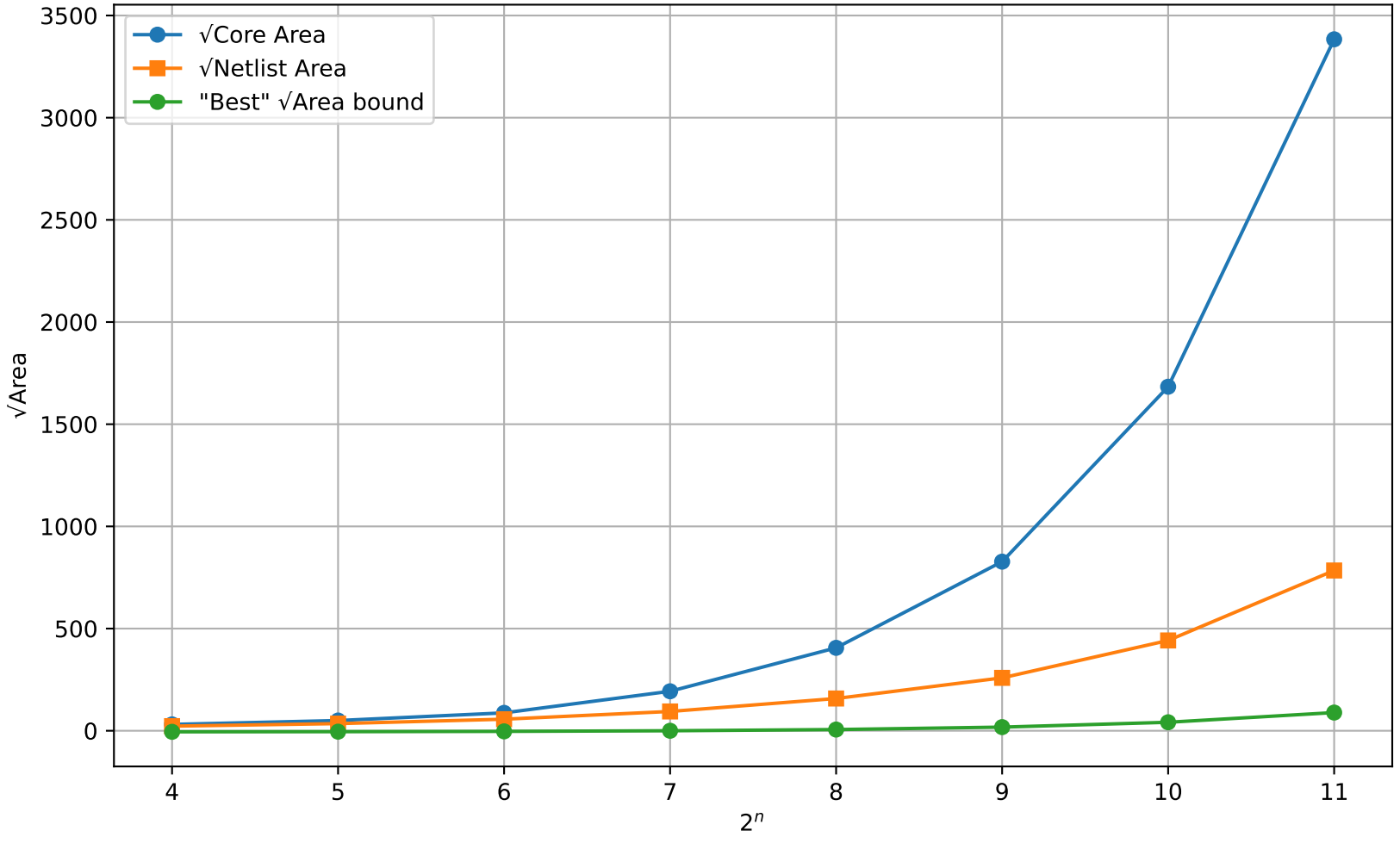
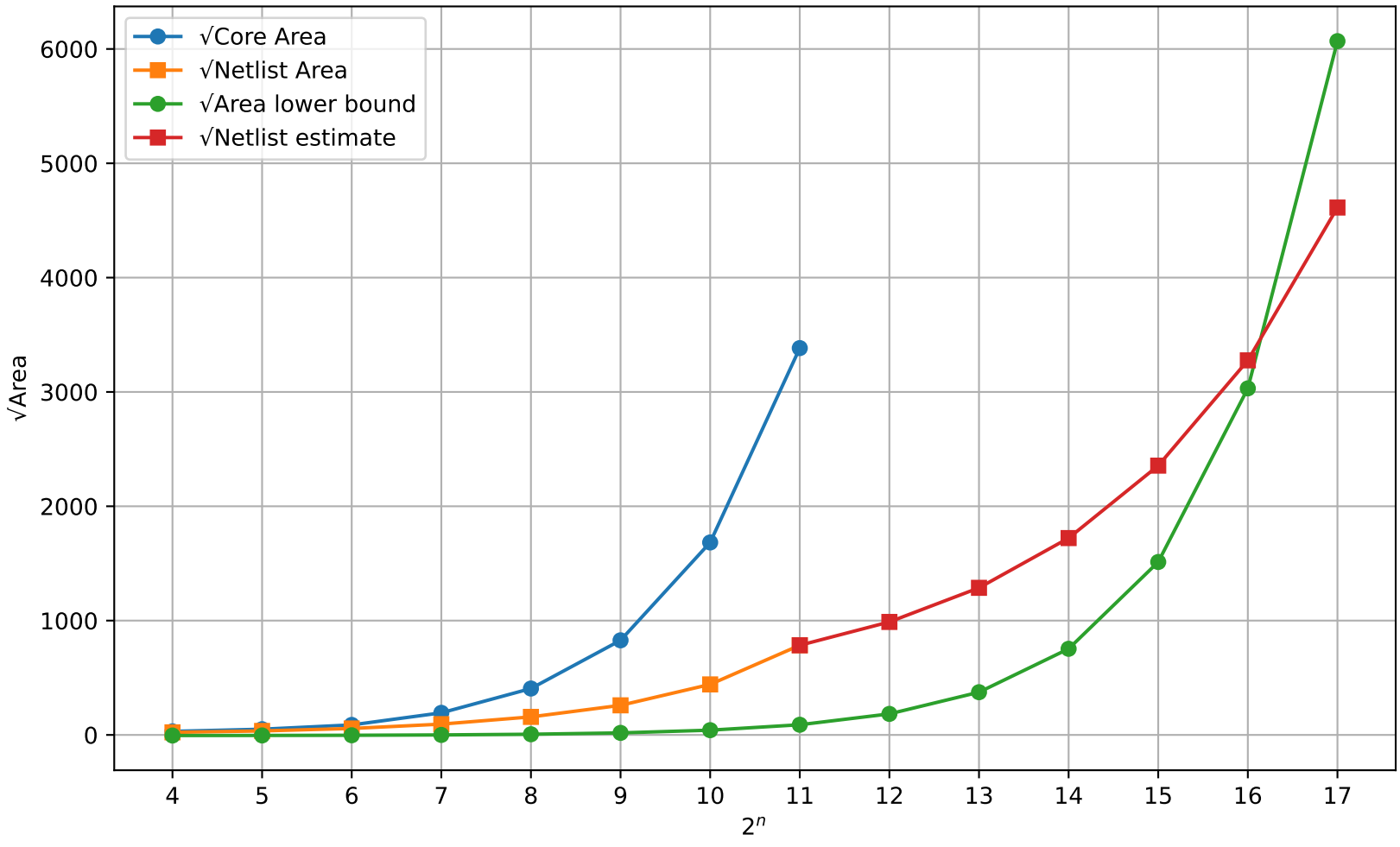
General Case	Canonical Partitions
$\sqrt{A} \cdot T \geq \Omega\left(\frac{N}{log_2(2N)}\right)$	$\sqrt{A} \cdot T \geq \Omega(N)$

Comparison

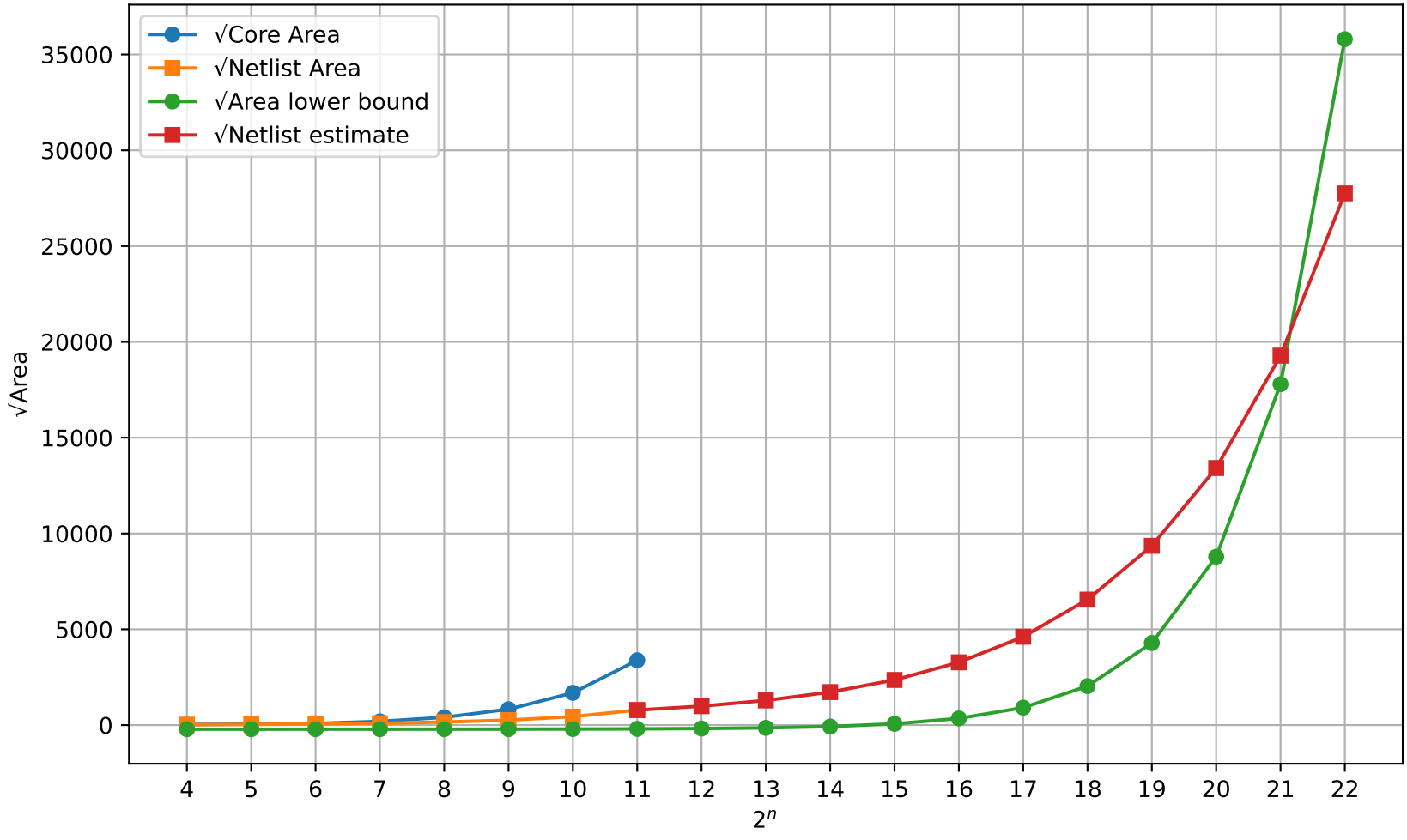
A moment of truth: Canonical partition



Not so gloomy: if inputs were aligned (smaller cut)



Removing assumption on inputs alignment but still canonical



Trugarez!