Impact of quantum computer on Impagliazzo's five worlds

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Work In Progress

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The plan

1. Classical Assumptions
2. Quantum Assumptions
3. Our result
4. High level idea of the proof
5. Conclusion

Minimal assumption for classical cryptography

One-Way Functions

A function $F : \{0,1\}^n \to \{0,1\}^n$ is a One-Way Function (OWF) if:

1. $F(x)$ can be computed efficiently.
2. Given $y = F(x)$, it is hard to compute x.

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One-Way Functions are minimal for cryptography

- Most advanced cryptographic schemes require one-way functions.
- For example, a hash function has to be a one-way function.
- There is nothing (interesting) weaker.

Some results about classical cryptography

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$$\exists OWF \Rightarrow \mathrm{P} \neq \mathrm{NP}$$

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Theorem ([IR89])

$$\exists OWF \not\Rightarrow \exists PKE$$

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Black-box impossibility results

A black-box impossibility result of A from B consists of exhibiting an oracle $\mathcal{O}$ such that, relative to $\mathcal{O}$, B exists but not A.

But. . .why? 🤔

Motivation

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No practical use

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No practical use
The goal is to understand the strength of assumptions and primitives

Worlds of cryptography

Impagliazzo's five worlds [Imp95]

- **Algorithmica** $\mathsf{P} = \mathsf{NP}$.
- **Heuristica** $\mathsf{P} \neq \mathsf{NP}$ but NP problems are easy on average.
- **Pessiland** $\mathsf{P} \neq \mathsf{NP}$ but one-way functions do not exist.
- **Minicrypt** One-way functions exist, but public key cryptography is impossible.
- **Cryptomania** Public key cryptography is possible.

Quantum Computation

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An algorithm can be written:

$$c_1 \quad c_2 \quad = \quad M \quad U_s \quad \cdots \quad U_0 \quad |\psi\rangle \quad |0\rangle^{\otimes m}$$

- c_1: Output
- c_2: Discarded bits
- M: Measurement
- U_s: Unitary
- U_0: Unitary
- $|\psi\rangle$: Input
- $|0\rangle^{\otimes m}$: Ancillas

Quantum Pseudorandomness

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Pseudorandom Number Generator

A function $F : \{0, 1\}^n \to \{0, 1\}^\ell$ is a Pseudorandom Number Generator (PRNG) if:

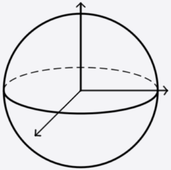
1. $F(x)$ can be computed efficiently.
2. $F(x) \approx \mathcal{U}_\ell$, when $x \leftarrow \mathcal{U}_n$.
3. $\ell > n$.

Quantum Pseudorandomness

Quantum Randomness

We can also consider quantum randomness.

The equivalent to the uniform distribution is the *Haar measure* μ_{2^n}.



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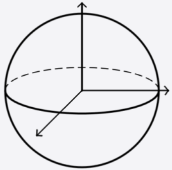

Pseudorandom Quantum States Generators

A function $F : \{0,1\}^\lambda \to \left(\mathbb{C}^2\right)^{\otimes n}$ is a Pseudorandom Quantum State generator (PRS) if:

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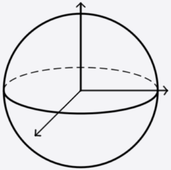

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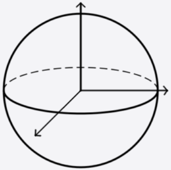

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Formal definition of PRSs

Definition (Pseudorandom quantum states [JLS18])

A keyed family of n-qubit quantum states $\{|\varphi_k\rangle\}_{k\in\{0,1\}^\lambda}$ is *pseudorandom* if the following two conditions hold:

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1. **Efficient generation**. There is a QPT algorithm G such that:

$$G_\lambda(k) = |\varphi_k\rangle \,.$$

2. **Pseudorandomness**. For any QPT adversary $\mathcal{A}$ and all polynomials $t(\cdot)$, we have:

$$\left| \Pr_{k\leftarrow\{0,1\}^\lambda}\left[\mathcal{A}\left(|\varphi_k\rangle^{\otimes t(\lambda)}\right)=1\right] - \Pr_{|\nu\rangle\leftarrow\mu_{2^n}}\left[\mathcal{A}\left(|\nu\rangle^{\otimes t(\lambda)}\right)=1\right]\right| \leq \mathrm{negl}(\lambda).$$

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Worlds relative to which quantum computation is possible.

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Worlds relative to which quantum computation is possible.

- Quantum Cryptomania: Public Key Cryptography exists! (resistant to quantum attacks)
- MiniQcrypt: Quantum resistant One-Way Functions exist!
- MicroCrypt: PRSs exist!

  oblivious transfer, multi party computation, public key encryption with quantum keys, quantum one-time digital signatures, pseudo one-time pad encryption schemes, statistically binding and computationally hiding commitments and quantum computational zero knowledge proofs, bit commitments...

Relation between quantum primitives

Theorem ([JLS18])

Classical 🎲 $\Rightarrow$ *Quantum* 🎲

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There can be quantum cryptography even if "P $=$ NP"

🤯

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PRU, PRFS, PRS, 1PRS, EFI pairs, OWSG...

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In the quantum setting however...

The Landscape of Quantum Assumptions

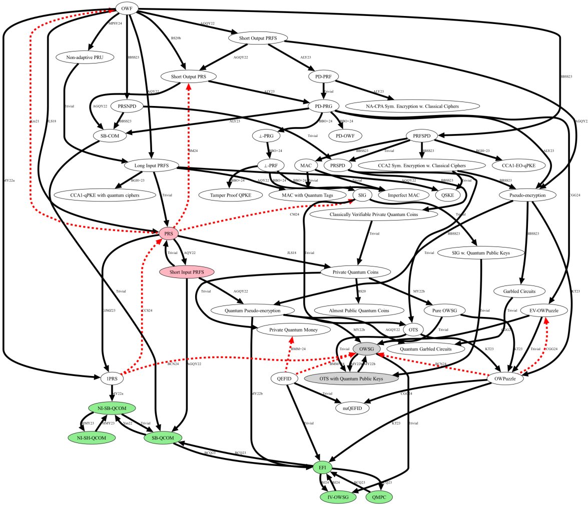

Figure: https://sattath.github.io/microcrypt-zoo/

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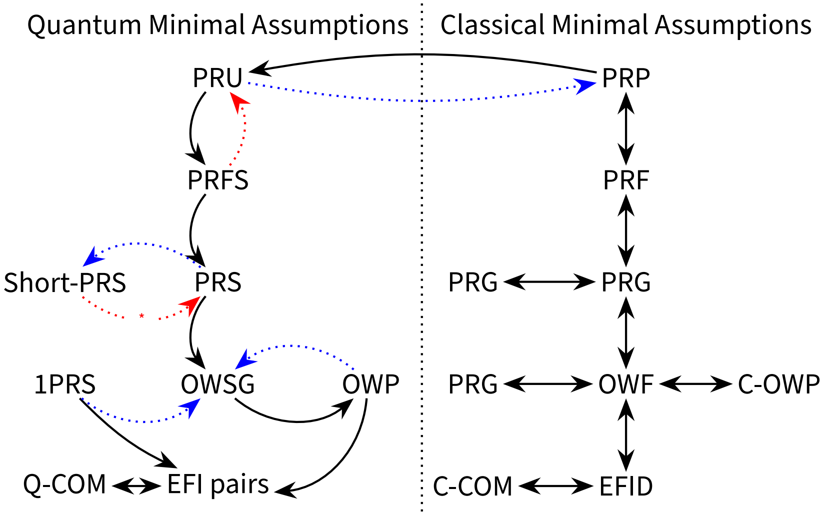

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- PRFSs are a natural generalization of PRSs.
- PRUs are *unitaries* that are indistinguishable from Haar-random *unitaries*.

$$\mathcal{O} = (\quad \mathcal{O}_1 \quad , \quad \mathcal{O}_2 \quad)$$

$\exists$ PRFS (for $\mathcal{O}_1$) $\quad\quad \nexists$ PRU (for $\mathcal{O}_2$)

Common Haar Function-like State Oracle

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Now let's rule out PRUs!

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2. **Pseudorandomness.** For any QPT adversary $\mathcal{A}$, we have:

$$\left| \Pr_{k\leftarrow\{0,1\}^\lambda}\left[\mathcal{A}^{U_k}\left(1^\lambda\right) = 1\right] - \Pr_{V\leftarrow\mu_{2^n}}\left[\mathcal{A}^{V}\left(1^\lambda\right) = 1\right]\right| \leq \mathrm{negl}(\lambda).$$

By contradiction, consider the PRU algorithm $\{G_k\}_{k \in \{0,1\}^*}$:

$$G_k : |\psi\rangle \mapsto |\psi'\rangle = U_T^{(k)} \; S_{x_T^{(k)}} \; \cdots \; U_1^{(k)} \; S_{x_1^{(k)}} \; U_0^{(k)} \; |\psi\rangle$$

Unitary — Query — Unitary — Query — Unitary

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Unitary Query Unitary Query Unitary

We assume there is no ancilla. (general case is WIP) 😬

The tools

Lemma (Swap test)

The swap test on input $(|\sigma\rangle, |\rho\rangle)$ outputs 1 with probability

$$\frac{1 + |\langle\rho|\sigma\rangle|^2}{2},$$

in which case we say that it passes the swap test.

Attack idea

The algorithm

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For all k:

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- But here, $\mathcal{O}_2$ will be dependant of $\mathcal{O}_1$, which is bad for the existence of PRFS! 😟
- We will approximate G_k without querying $\mathcal{O}_1$, and $\mathcal{O}_2$ will be independent of $\mathcal{O}_1$ 😌

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Therefore, one may argue that

$$\begin{aligned} G_k |\psi\rangle &= U_T^{(k)} \cdot S_{x_T^{(k)}} \cdot U_{T-1}^{(k)} \cdot \ldots \cdot U_1^{(k)} \cdot S_{x_1^{(k)}} \cdot U_0^{(k)} |\psi\rangle \\ &\approx U_T^{(k)} \cdot U_{T-1}^{(k)} \cdot \ldots \cdot U_1^{(k)} \cdot U_0^{(k)} |\psi\rangle . \end{aligned}$$

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$$\approx U_T^{(k)} \cdot U_{T-1}^{(k)} \cdot \ldots \cdot U_1^{(k)} \cdot U_0^{(k)} |\psi\rangle .$$

However, the loss is proportional to $1/2^{|x|}$ 🤓👆

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$$\tilde{S}_x = \begin{cases} S'_x, & \text{for small } |x|, \\ I, & \text{for large } |x|. \end{cases}$$

$$F_k : |\psi\rangle \mapsto U_T^{(k)} \cdot \tilde{S}_{x_T^{(k)}} \cdot U_{T-1}^{(k)} \cdot \ldots \cdot U_1^{(k)} \cdot \tilde{S}_{x_1^{(k)}} \cdot U_0^{(k)} |\psi\rangle ,$$

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The attack

Prepares $\Phi = (|\rho\rangle \otimes V|\rho\rangle)^{\otimes M}$ for some large M and defines:

P_k: on input $\Phi = (|\rho\rangle \otimes V|\rho\rangle)^{\otimes M}$, it applies $(F_k \otimes \mathrm{Id})^{\otimes M}$, applies M swap tests on each copy; if sufficiently many copies pass the swap test, it returns 1. Otherwise it returns 0.

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We can show that

- if $V = G_k$, P_k returns 1 with high probability,
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This can be done with a QPSPACE oracle! (Quantum OR Lemma) 🔥
Relative to $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$ =(CHFS,QPSPACE), we have PRUs but not PRFSs!

Other results

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$$\exists \text{short-PRFS} \not\Rightarrow \exists\ PRG \quad \textit{(with negligible correctness)}$$

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This complements a previous result that shows that there exits an oracle relative which PRSs exist but short PRSs do not.

Conclusion

- Oracle separation of PRUs and PRFSs (There is still some work left to finish our proof!)
- Conditionned oracle separation of short-PRSs and PRSs.
- Also, there is still a lot left to do to fully grasp the strength of quantum assumptions.

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- Conditionned oracle separation of short-PRSs and PRSs.
- Also, there is still a lot left to do to fully grasp the strength of quantum assumptions.

Thank you for your attention!

Lemma (Quantum OR lemma)

Let $\{\Pi_i\}_{i\in[N]}$ be POVMs. Let $0 < \varepsilon < 1/2$ and $\delta > 0$. Let Ψ be a quantum state such that either

1. *there exists $i \in [N]$ such that $\mathrm{Tr}[\Pi_i \Psi] \geq 1 - \varepsilon$, or*
2. *for all $i \in [N]$, $\mathrm{Tr}[\Pi_i \Psi] \leq \delta$.*

Then, there is a quantum circuit C, such that in case i)

$$\Pr(1 \leftarrow C(\Psi)) \geq \frac{(1-\varepsilon)^2}{7},$$

and in case ii),

$$\Pr(1 \leftarrow C(\Psi)) \leq 4N\delta.$$

The circuit C can be implemented in QPSPACE.

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