Variations on the Knapsack Generator

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KNAPSACK GENERATOR

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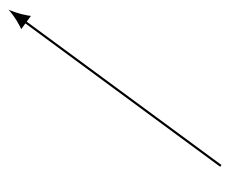

Pseudo Random Number Generator


KNAPSACK
GENERATOR
Hard computational problem
Pseudo Random Number Generator

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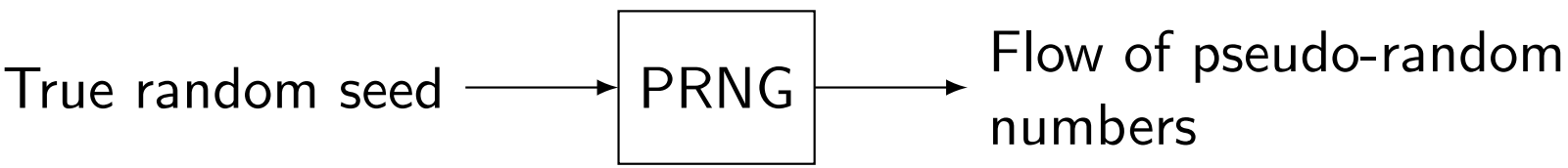

Security of PRNG

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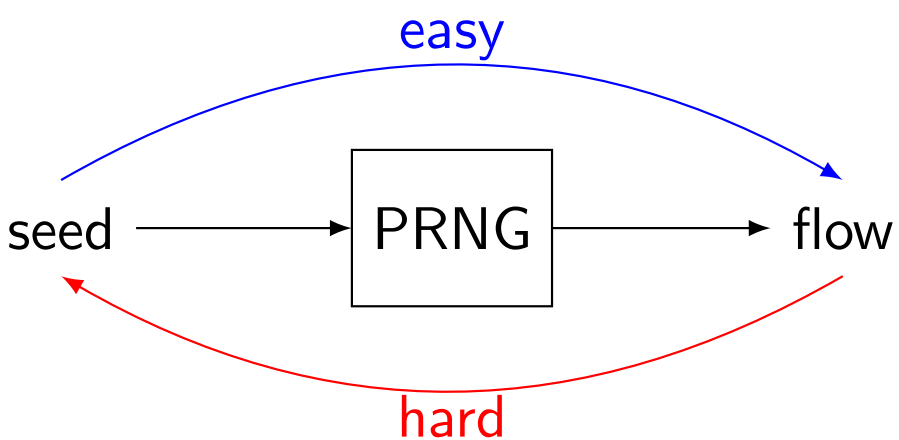
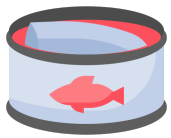
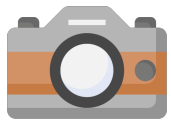


(almost) Knapsack Problem

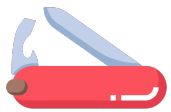
t kg



ω_1 kg



ω_2 kg



ω_3 kg

ω_4 kg

What is in the knapsack ?

Mathematic version

The weight list:	$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n) \in \{0, N\}^n$
The secret composition:	$\mathbf{u} = (u_1, \ldots, u_n) \in \{0, 1\}^n$
The target weight:	$v = \sum \omega_i u_i = \langle \boldsymbol{\omega}, \mathbf{u} \rangle$

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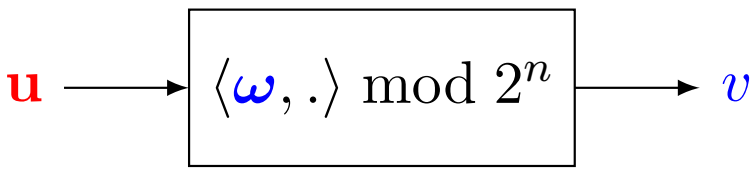
The Subset Sum Problem is NP-hard and remain hard if we replace v by $v \mod N$ as long as $N \simeq 2^n$.

Knapsack Generator by Rueppel and Massey[1]



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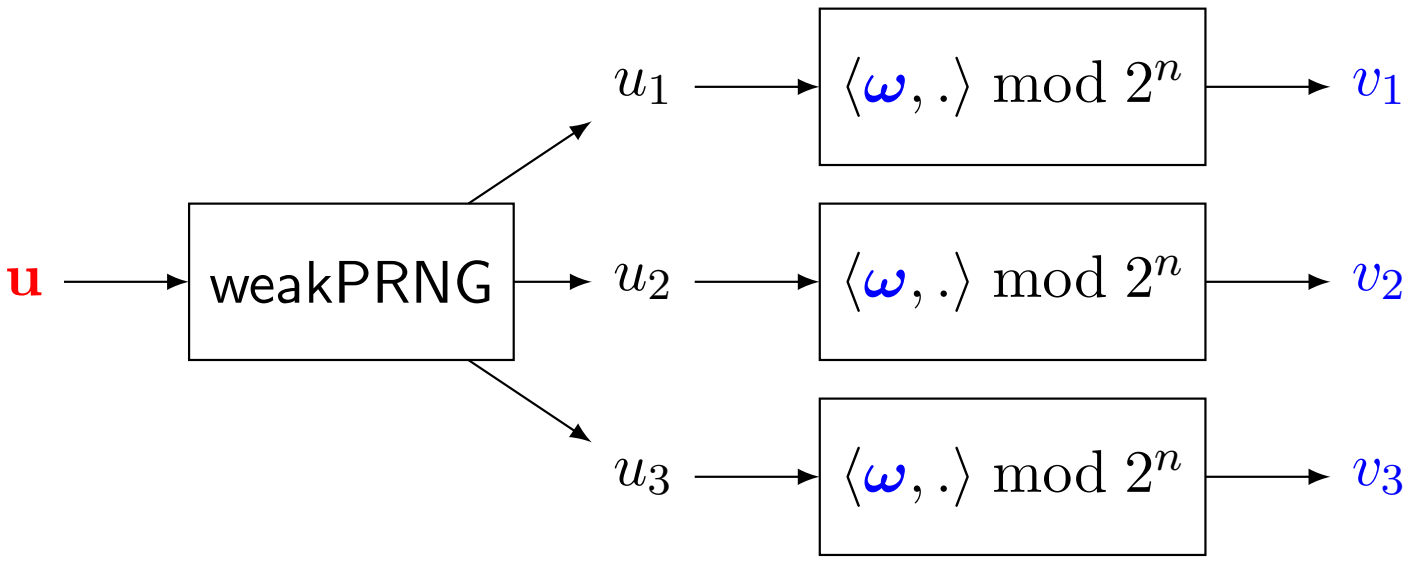

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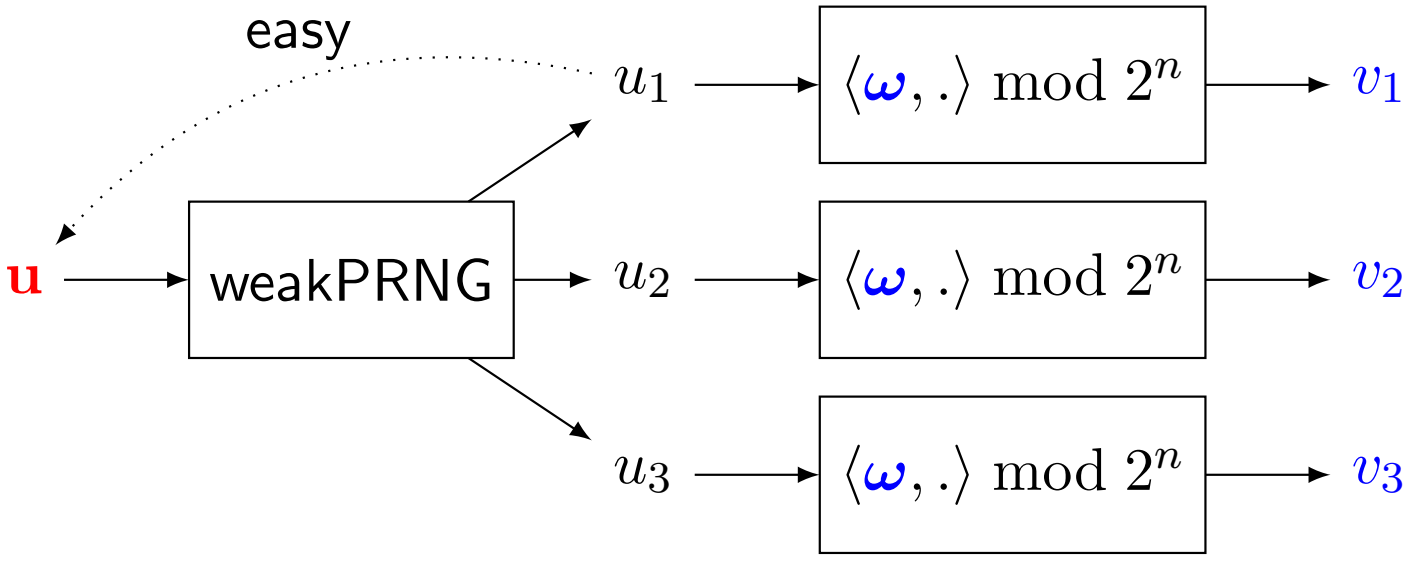

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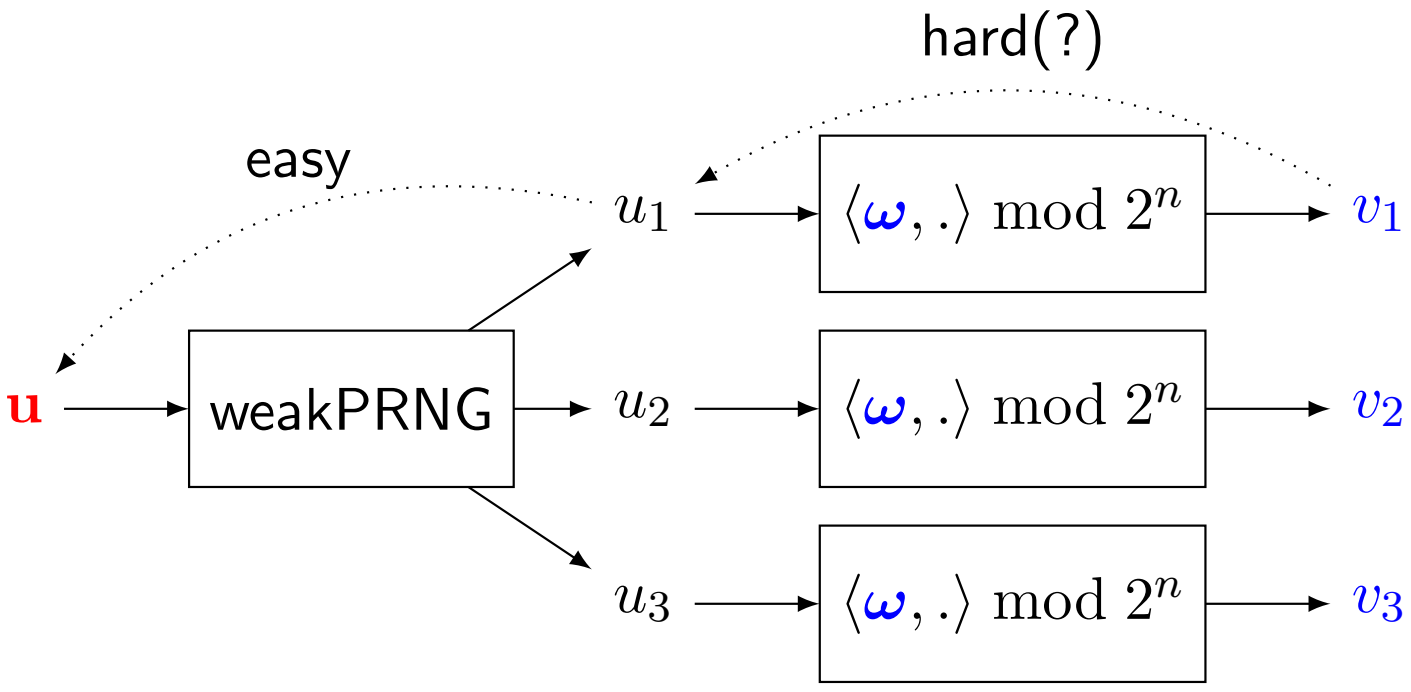

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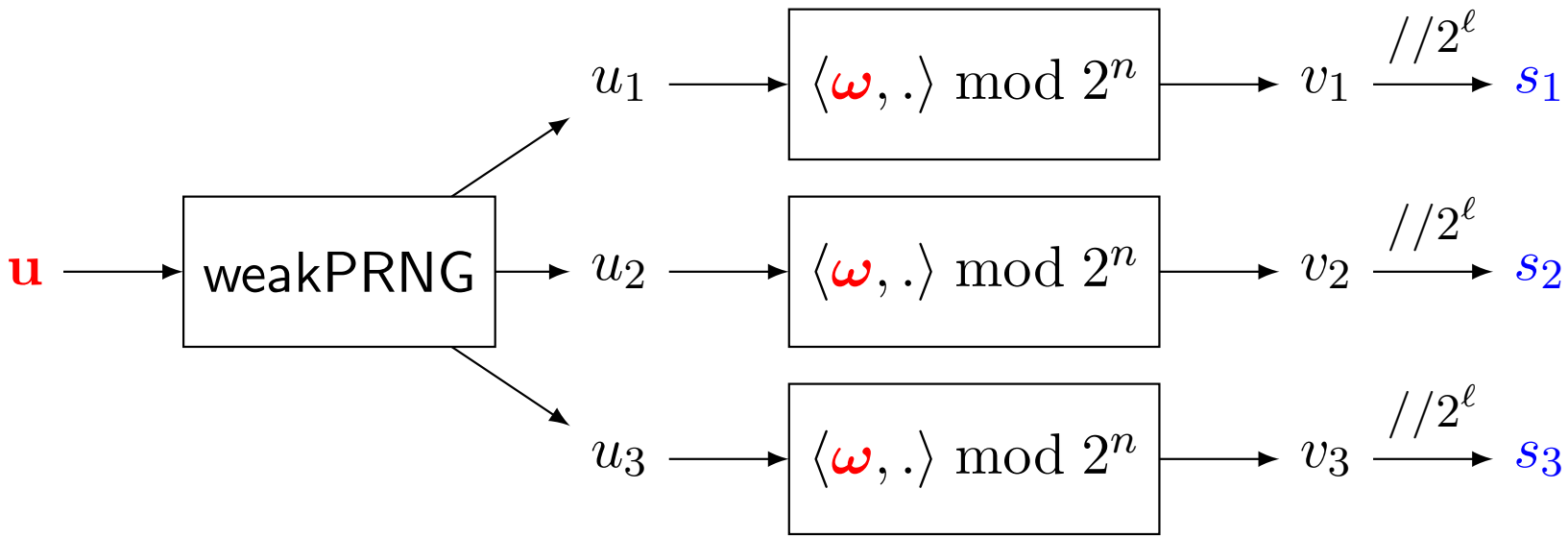

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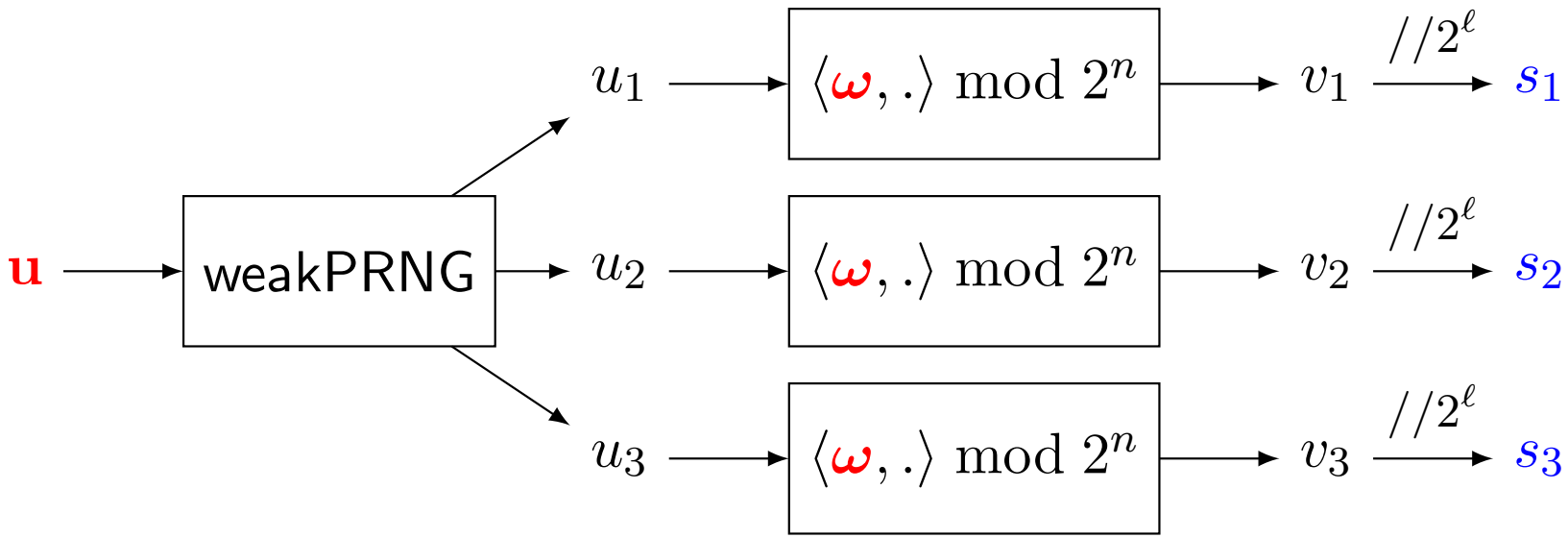

We call δ_i the truncated bits : $v_i = 2^\ell s_i + \delta_i$.

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$$\text{secret} : \quad u \quad + \quad \omega$$

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n bits n^2 bits

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Can we distinguish between the u ?
Yes, with OmegaRetriever

Distinguish between u

We consider m outputs and $\mathbf{s} = (s_1, \ldots, s_m)$.

OmegaRetriever:	$\mathbf{u}, \mathbf{s}$	$\rightarrow \boldsymbol{\omega}'$	close to $\boldsymbol{\omega}$
	$\mathbf{u}', \mathbf{s}$	$\rightarrow \boldsymbol{\omega}''$	not close to $\boldsymbol{\omega}$

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KnapsackGen$(u, \boldsymbol{\omega}')$ will be close to KnapsackGen$(u, \boldsymbol{\omega})$.
KnapsackGen$(u', \boldsymbol{\omega}'')$ will be not.

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$$\boxed{\boldsymbol{\omega} = T2^\ell \mathbf{s} + T\boldsymbol{\delta}}$$

OmegaRetriever from FSE 2011 (part 2)

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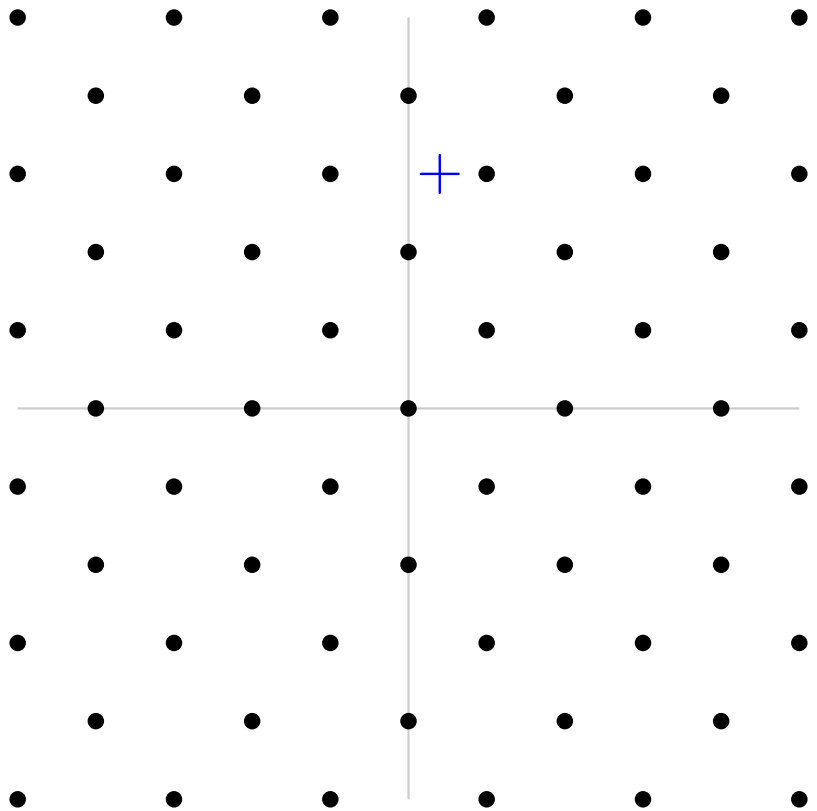
Experimental results are close to the bound.

Closest Vector Problem



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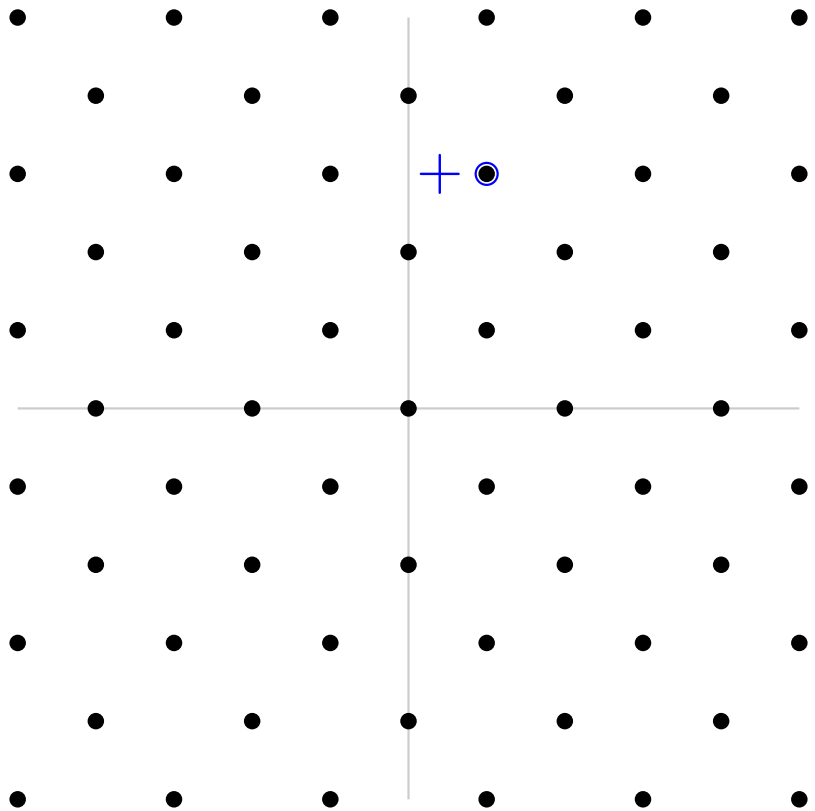

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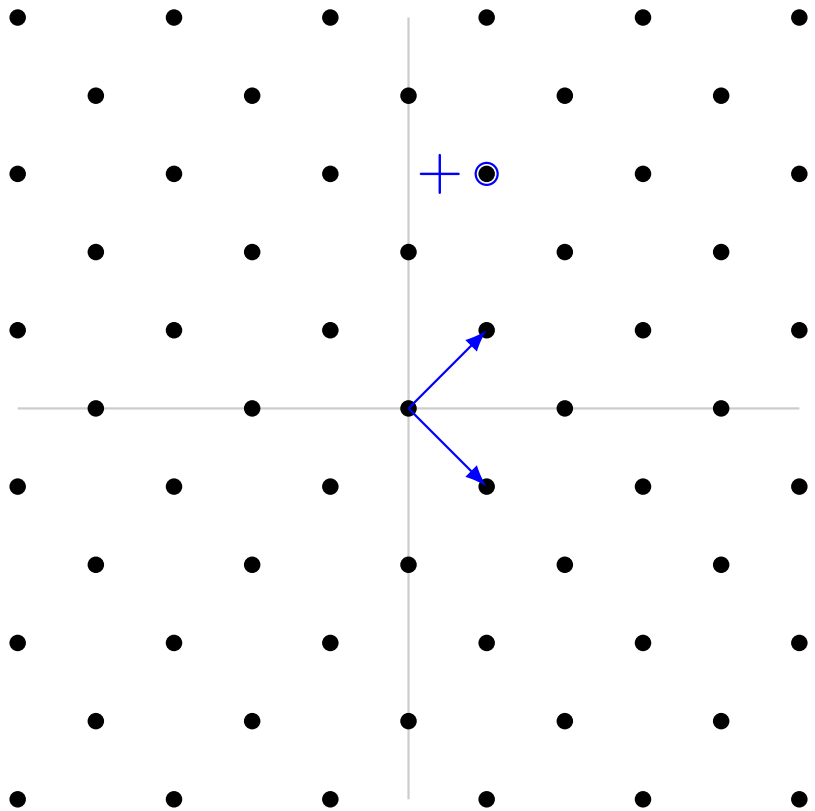

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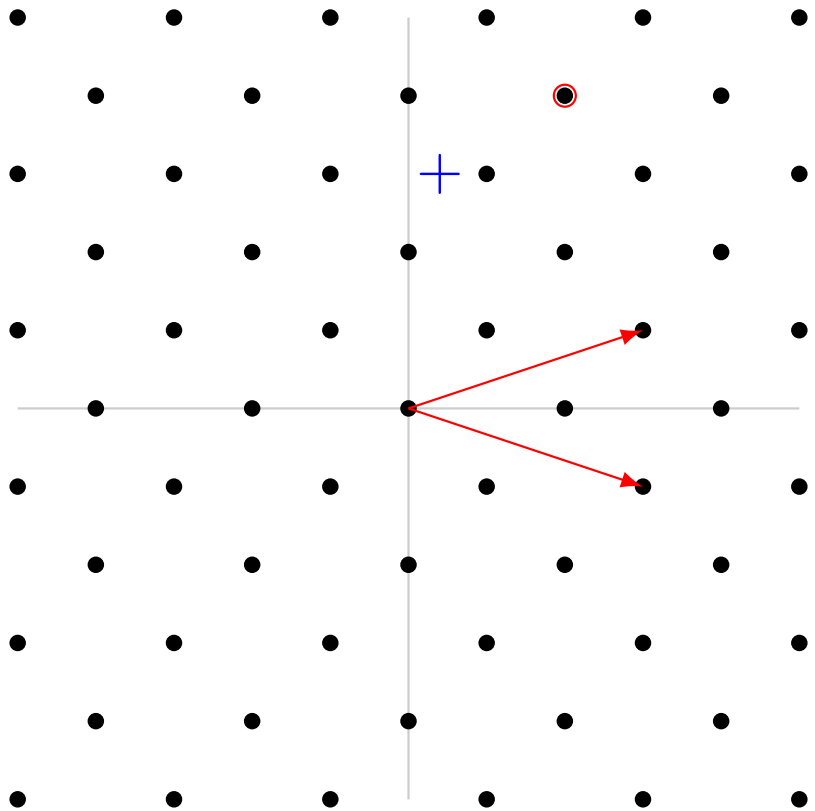

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But $\boldsymbol{\omega}'$ defined as $U \times \boldsymbol{\omega}' \equiv \mathbf{v}' \bmod 2^n$ is close to $\boldsymbol{\omega}$!

Why does it work ?

- $\mathbf{v} - \mathbf{v}'$ is small and equal to $U \times (\boldsymbol{\omega} - \boldsymbol{\omega}') \bmod 2^n$
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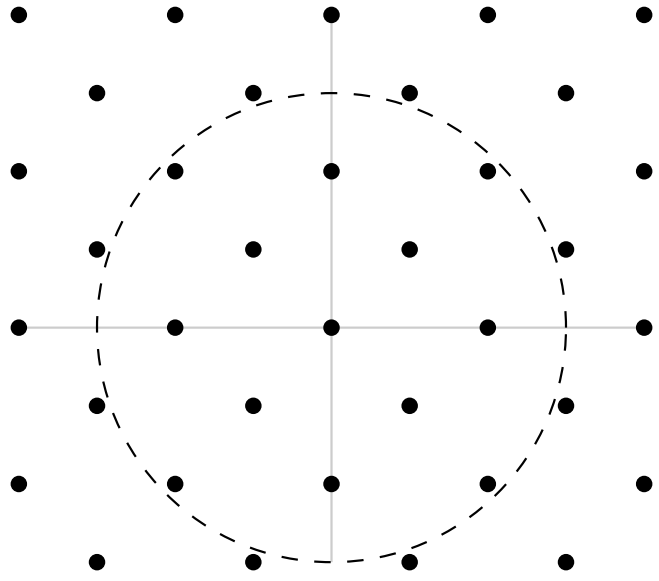
- BUT $\|\boldsymbol{\omega} - \boldsymbol{\omega}'\| \ll \|T\| \times \|(\mathbf{v} - \mathbf{v}')\|$

A Second Idea

1. We know that $\mathbf{v} - \mathbf{v}'$ is small ($\leq K$) and in Λ.



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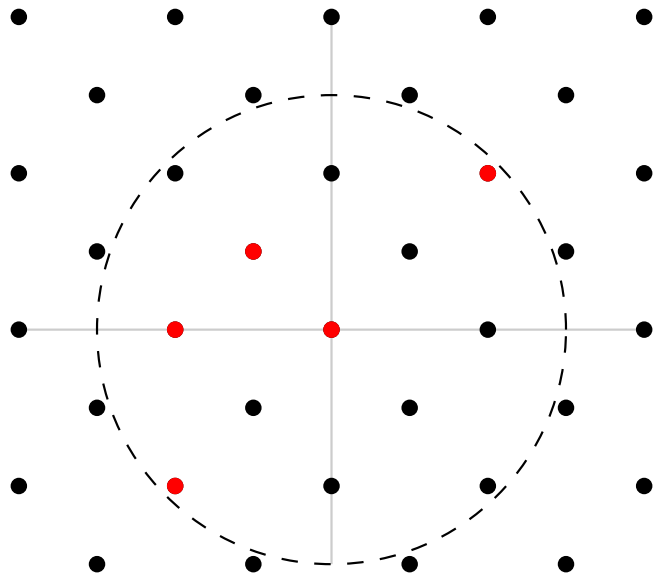

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1. We know that $\mathbf{v} - \mathbf{v}'$ is small ($\leq K$) and in Λ.
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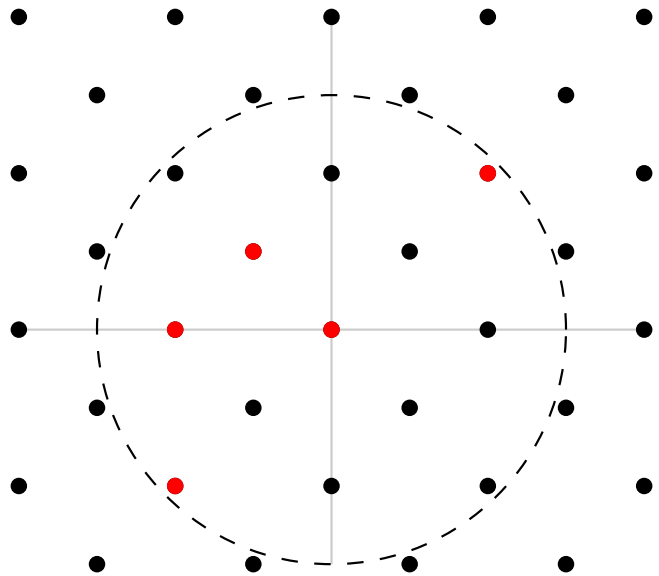

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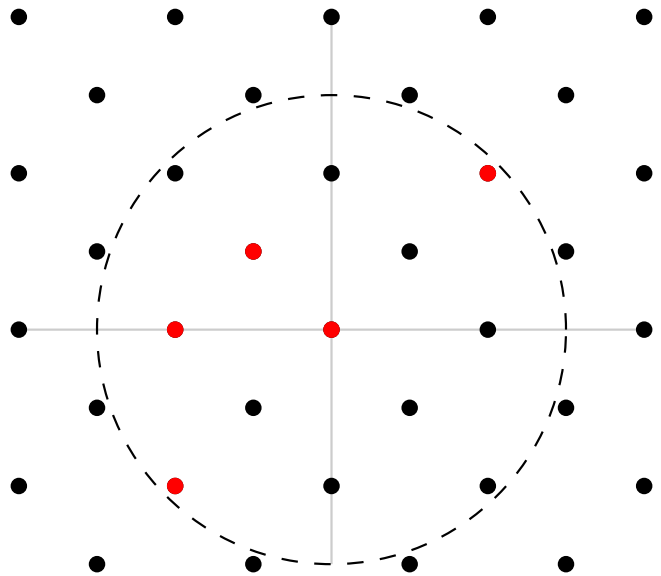

We denote A_K the set of red points

$$|A_K| = (2 \times \lfloor K/\|U\| \rfloor - 1)^n$$

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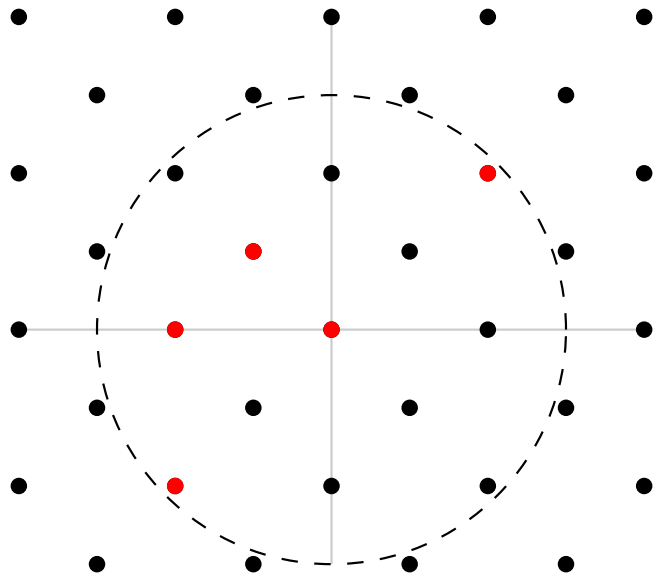

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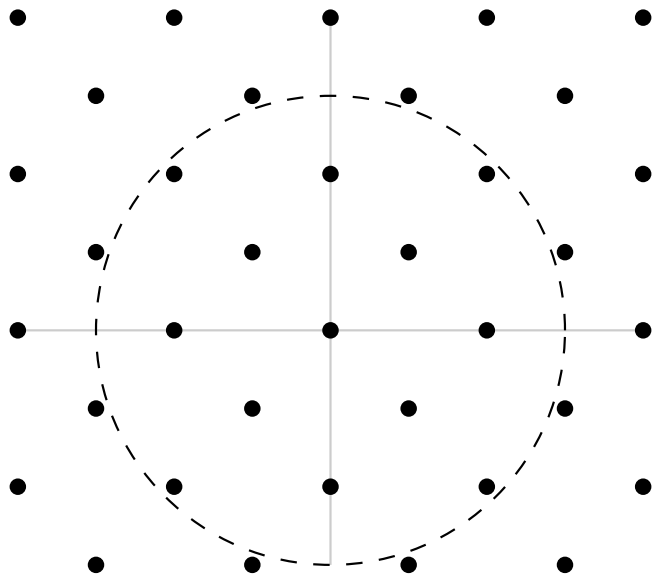
We denote B_K the set of points in the ball

How many point in B_K ?



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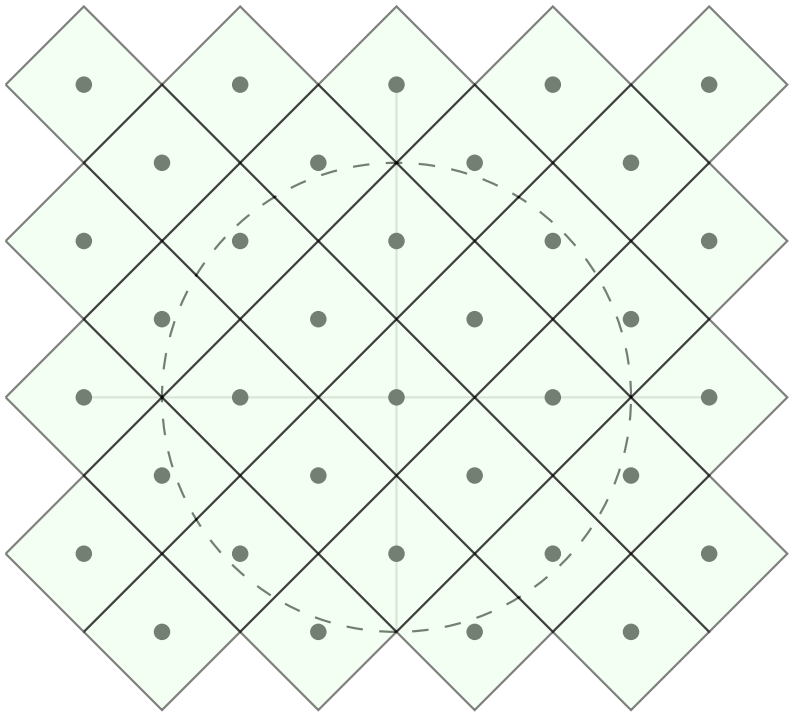

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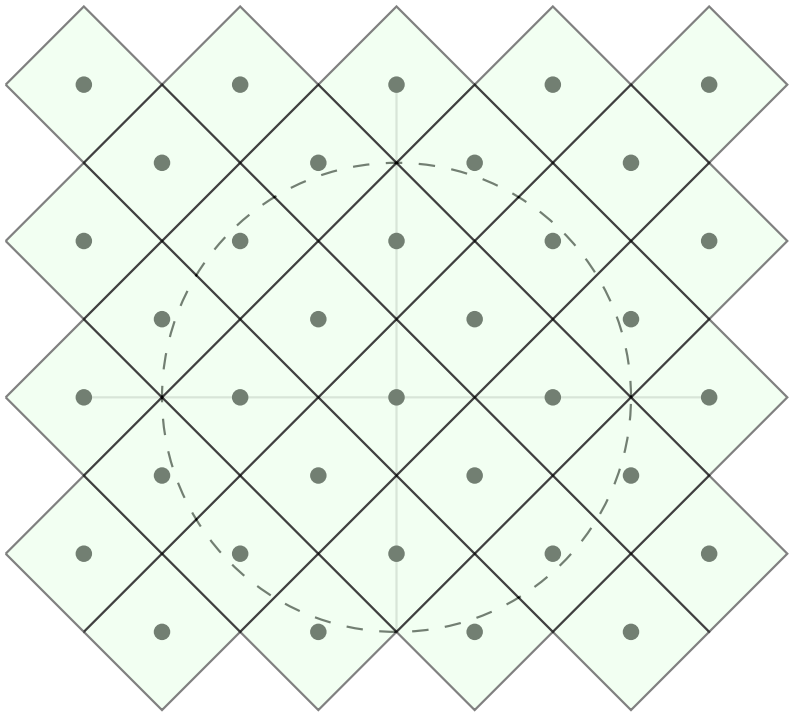
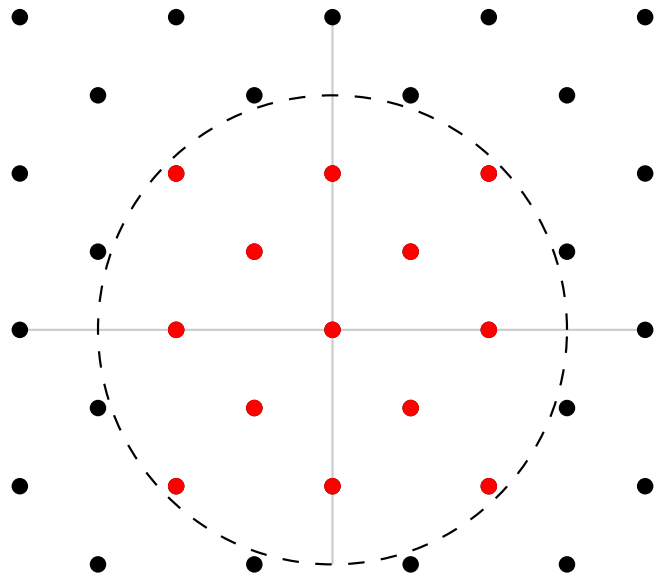


Gaussian Heuristic : $|B_K| \simeq Volume(Ball)/Volume(\Lambda)$

In the case where $n = 32$, $m = 42$ and $\ell \leq 15$,

$$|A_K| \geq |B_K| \text{ with } K = 2^{\ell+1}$$



Thus $\mathbf{v} - \mathbf{v}'$ is a red point and $\|\boldsymbol{\omega} - \boldsymbol{\omega}'\| < K/\|U\|$.

ℓ	5		10		15		20		25	
m	34	40	34	40	34	40	35	40	39	40
✓bits (over 32)	27	28	22	23	5	18	4	13	6	8

Figure: Quality of $\boldsymbol{\omega'}$ for $n = 32$

ℓ	5		10		15		20
m	34	40	35	40	36	40	41
✓bits (over 32)	10	22	10	17	8	12	6

Figure: Quality of $\boldsymbol{\omega'}$ for $n = 32$ for FSE 2011 algorithm

Thank you for your attention,